Backtesting Marginal Expected Shortfall and Related Systemic Risk Measures

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Systemic risk

- The recent financial crisis has fostered extensive research on systemic risk, either on its definition, measurement, or regulation (Bisias et al. 2012, Benoit et al. 2016).
- The ultimate goal of this research is to better identifying the vulnerabilities of the financial system.
- Ideally, regulators need **measures of systemic risk** that are timely, capture well-identified economic mechanisms, and can be used as an input for regulatory tools.

In practice, measuring the contribution of a given firm to the overall risk of the system is challenging.

1. A first approach relies on **structural models** that identify specific sources of systemic risk, such as contagion, bank runs, or liquidity crises.

2. The **regulatory approach** is based on **proprietary data** (cross-positions, size, leverage, liquidity, interconnectedness, etc.). *Ex: FSB-BCBS methodology used to identify the G-SIB.*

3. A third approach aims to derive global measures of systemic risk based on **market data**, such as stock or asset returns, option prices, or CDS spreads.

The most well-known market-based systemic risk measures are:

1. Marginal Expected Shortfall (**MES**) and the Systemic Expected Shortfall (**SES**) of Acharya et al. (2010)
2. the Systemic Risk Measure (**SRISK**) of Acharya et al. (AER, 2012) and Brownlees and Engle (2015),
3. Delta Conditional Value-at-Risk (Δ**CoVaR**) of Adrian and Brunnermeier (AER, 2016).

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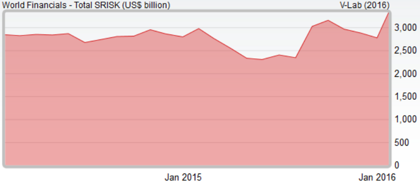
US Top 10 SRISK	SRISK	MES	LVG
Bank of America Corp	72,233	2.78	14.37
Citigroup Inc	65,886	4.00	13.33
JPMorgan Chase & Co	63,507	3.42	10.98
Prudential Financial Inc	41,005	3.35	23.49
Morgan Stanley	39,539	4.29	16.28
MetLife Inc	38,706	3.03	17.52
Goldman Sachs Group Inc/The	29,652	3.46	12.58
Lincoln National Corp	14,394	3.96	25.00
Principal Financial Group Inc	11,211	3.73	19.45
Hartford Financial Services Group Inc/The	8,820	3.17	14.45

World Financials - Total SRISK (US$ billion)   V-Lab (2016)



Example (EBA Stress test, October 26, 2014)

Twenty-four european banks fail EBA stress test. All the French banks succeeded the tests.

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Last updated: October 27, 2014 10:38 pm

Alternative stress tests find French banks are weakest in Europe

Tom Braithwaite

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Measure based on market capitalisation says they are most risky

On Sunday, Christian Noyer, governor of the Banque de France, was crowing about the "excellent" performance of French banks on the European stress tests

Many of their Italian and Greek counterparts might have flunked but France could be proud of its banking sector. "The French banks are in the best positions in the eurozone," said Mr Noyer.

Not so fast.

Two days earlier, a different test found that the French financial sector was the weakest in Europe.

The team with the temerity to deliver this bucket of cold water to Paris works at the wonderfully named Volatility Institute at New York University's Stern school and presented its findings from a safe distance – a financial conference at the University of Michigan.

The chief architect, Viral Acharya has worked on systemic risk ever since the last crisis, attempting to design a bank safety test that can be run all the time – not at the whim of regulators.

Using his methodology, which he calls SRISK Mr Acharya found that in a crisis French financial institutions would have a capital shortfall of almost $400bn, worse than the US and UK despite their much bigger financial sectors. Looking just at the French banks tested in the ECB stress tests, which found zero capital shortfall, SRISK came up with €189bn.

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Validation

- **Theoretical approach**: Chen, Iyengar, and Moallemi (2013) define an axiomatic framework for systemic risk measures ("coherent" systemic risk measure).
- **Empirical assessment**:
  - Brownlees and Engle (2015) show that banks with higher SRISK before the financial crisis were more likely to receive capital injections from the Federal Reserve.
  - To test whether firms with high systemic risk scores are more likely to become insolvent (Wu and Zhao 2014), to suffer the highest financial losses (Idier, Lamé, and Mesonnier 2014) in a financial crisis.

Backtesting

- As defined by Jorion (2007), backtesting is a formal statistical framework that consists in verifying if actual losses are in line with projected losses.
- This involves a systemic comparison of the history of model-generated risk measure forecasts with actual returns.

Goal of the paper

1. Propose a backtesting procedure for the MES, similar to that used for the VaR (Kupiec, 1995, Christoffersen, 1998, etc.).
2. Take into account the estimation risk (Escanciano and Olmo 2010, 2011, Gouriéroux and Zakoian, 2013)
3. To generalize the backtesting procedure to the MES-based systemic risk measures (SES, SRISK) and to the ΔCoVaR.

Outline

1. Marginal Expected Shortfall (MES)
2. Cumulative joint violation process
3. Backtesting MES
4. Monte Carlo simulation study
5. Backtesting other systemic risk measures
6. Conclusion

Notations:

- $Y_t = (Y_{1t}, Y_{2t})'$ denote a vector of stock returns for two assets at time t.
- Y_{1t} generally corresponds to the stock return of a financial institution, whereas Y_{2t} corresponds to the market return.
- Ω_{t-1} is the information set available at time $t-1$
- $F(.;\Omega_{t-1})$ is the *cdf* of Y_t given Ω_{t-1} such that $\forall y = (y_1, y_2)' \in \mathbb{R}^2$

$$F(y_t;\Omega_{t-1}) \equiv \Pr\left(Y_{1t} < y_1, Y_{2t} < y_2 \middle| \Omega_{t-1}\right)$$

Definition (MES, Acharya et al. 2010)

The MES of a financial firm is the short-run expected equity loss conditional on the market taking a loss greater than its VaR:

$$MES_{1t}(\alpha) = \mathbb{E}(Y_{1t} | Y_{2t} \leq VaR_{2t}(\alpha); \Omega_{t-1})$$

where $VaR_{2t}(\alpha)$ denotes the α-level VaR of Y_{2t}, such that $\Pr(Y_{2t} \leq VaR_{2t}(\alpha) | \Omega_{t-1}) = \alpha$ with $\alpha \in [0, 1]$

Remark

- The **MES** can be expressed as a function of the quantiles of the conditional distribution of Y_{1t} given $Y_{2t} \leq VaR_{2t}(\alpha)$
- These quantiles are nothing else than **Conditional VaR (CoVaR)** (Adrian and Brunnermeier, 2016).

Definition (CoVaR)

For any coverage level $\beta \in [0,1]$, the CoVaR for the firm 1 is the quantity $CoVaR_{1t}(\beta,\alpha)$ such that

$$\Pr\left(Y_{1t} \leq CoVaR_{1t}(\beta,\alpha) \middle| Y_{2t} \leq VaR_{2t}(\alpha);\Omega_{t-1}\right) = \beta.$$

The (β,α)-level CoVaR can also be defined as

$$CoVaR_{1t}(\beta,\alpha) = F^{-1}_{Y_1|Y_2 \leq VaR_{2t}(\alpha)}(\beta;\Omega_{t-1}),$$

Lemma (MES - CoVaR)

Definition of cond. probability and a change in variables yields

$$MES_{1t}(\alpha) = \int_0^1 CoVaR_{1t}(\beta, \alpha) \mathrm{d}\beta.$$

Risk Model

In general, the MES forecasts are issued from a parametric model specified by the researcher, the risk manager or the regulator.

- θ_0 denotes an unknown model parameter in $\Theta \in \mathbb{R}^p$
- $F\left(y;\Omega_{t-1},\theta_0\right)$ of the cdf of Y_t, the *cdf*
- $F_{Y_2}\left(.;\Omega_{t-1},\theta_0\right)$ of the marginal cdf of Y_{2t}
- $F_{Y_1|Y_2\leq VaR_{2t}(\alpha;\theta_0)}(y;\Omega_{t-1},\theta_0)$ the cdf of the truncated distribution of Y_{1t} given $Y_{2t} \leq VaR_{2t}(\alpha,\theta_0)$.

Backtesting

- The **backtesting** is a formal statistical framework that consists in verifying if actual losses are in line with projected losses (Jorion 2007).
- This involves a systemic comparison of the history of model-generated risk measure forecasts with actual returns.
- This comparison generally relies on tests over **violations**.

Example

A VaR violation is said to occur when the *ex-post* portfolio return is lower than the VaR forecast.

$$h_t(\alpha) = \mathbf{1}(Y_t \leq VaR_t(\alpha))$$

where $\mathbf{1}(.)$ denotes the indicator function.

Cumulative joint violation process

- In order to backtest the MES, we propose a **cumulative joint violation process**
- This cumulative joint violation can be viewed as a kind of violations "counterpart" of the MES definition

Du, Z. and J.C, Escanciano (2016): "Backtesting expected shortfall: Accounting for tail risk", forthcoming in Management Science.

Definition (joint violation)

We define a joint violation of the (β, α)-CoVaR of Y_{1t} and the α-VaR of Y_{2t}

$$
\begin{aligned}
h_t(\alpha, \beta, \theta_0) \quad = \quad & \mathbf{1}\left(Y_{1t} \leq CoVaR_{1t}\left(\beta, \alpha, \theta_0\right)\right) \times \\
& \times \mathbf{1}\left(Y_{2t} \leq VaR_{2t}\left(\alpha, \theta_0\right)\right)
\end{aligned}
$$

Remark: the centered violation $\{h_t(\alpha, \beta, \theta_0) - \alpha\beta\}_{t=1}^{\infty}$ is a *mds* $\forall\,(\alpha, \beta) \in [0,1]^2$.

$$\mathbb{E}\left(h_t(\alpha, \beta, \theta_0) - \alpha\beta \,\middle|\, \Omega_{t-1} \right) = 0$$

Definition (cumulative violation)

The cumulative *joint* violation process is defined as the integral of the violations $h_t(\alpha, \beta, \theta_0)$ for all the risk levels β between 0 and 1

$$H_t\left(\alpha, \theta_0\right) = \int_0^1 h_t(\alpha, \beta, \theta_0)\mathrm{d}\beta.$$

Remarks

1. The *mds* property of $\{h_t(\alpha, \beta, \theta_0) - \alpha\beta\}_{t=1}^{\infty}$ is preserved by integration

$$\mathbb{E}\left(H_t\left(\alpha, \theta_0\right) - \alpha/2 \middle| \Omega_{t-1}\right) = 0,$$

2. Furthermore

$$\mathbb{V}\left(H_t\left(\alpha, \theta_0\right) \middle| \Omega_{t-1}\right) = \alpha\left(1/3 - \alpha/4\right).$$

3. Finally, it is possible to rewrite $H_t\left(\alpha, \theta_0\right)$ in a more convenient way, through the Probability Integral Transformation (PIT)

Definition (generalized errors)

Define two "generalized" errors, namely:

$$u_{2t}\left(\theta_0\right) = F_{Y_2}(Y_{2t}; \Omega_{t-1}, \theta_0),$$

$$u_{12t}\left(\theta_0\right) = \frac{1}{\alpha} F\left(\widetilde{y}; \Omega_{t-1}, \theta_0\right),$$

where the vector $\widetilde{y}$ is defined as $\widetilde{y} = (y_1, VaR_{2t}(\alpha, \theta_0))'$.

Definition (cumulative violation)

The process $H_t(\alpha, \theta_0)$ can be expressed as a function of the "generalized errors" u_{2t} and u_{12t}, such as

$$H_t(\alpha, \theta_0) = (1 - u_{12t}(\theta_0)) \times \mathbf{1}(u_{2t}(\theta_0) \leq \alpha).$$

Backtesting MES

- Exploiting the *mds* property of the cumulative joint violation process, we propose two backtests for the MES.
- These tests are similar to those generally used by the regulator or the risk manager for VaR backtesting (Kupiec 1995, Christoffersen 1998, etc.).

Backtesting MES

1. The Unconditional Coverage (hereafter UC) test corresponds to the null hypothesis

$$H_{0,UC} : \mathbb{E}\left(H_t\left(\alpha, \theta_0\right)\right) = \alpha/2.$$

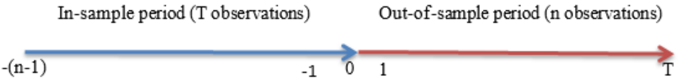
2. The null of the Independence test (IND) is defined as

$$H_{0,IND} : \rho_1 = ... = \rho_K = 0.$$

$$\rho_k = corr\left(H_t\left(\alpha, \theta_0\right) - \alpha/2, H_{t-k}\left(\alpha, \theta_0\right) - \alpha/2\right)$$

Estimation

These two tests imply to estimate the parameters $\theta_0 \in \Theta$. Denote by $\widehat{\theta}_T$ a consistent estimator of θ_0 (fixed forecasting scheme).



The backtesting tests are based on the out-of-sample forecasts of the cumulative violation process process given by

$$H_t(\alpha, \widehat{\theta}_T) = \left(1 - u_{12t}(\widehat{\theta}_T)\right) \mathbf{1}\left(u_{2t}(\widehat{\theta}_T) \leq \alpha\right), \quad \forall t = 1, ..., n.$$

Definition (UC test statistic)

The test statistic for UC, denoted UC_{MES}, is defined as

$$UC_{MES} = \frac{\sqrt{n}\left(\bar{H}(\alpha, \widehat{\theta}_T) - \alpha/2\right)}{\sqrt{\alpha\,(1/3 - \alpha/4)}},$$

with $\bar{H}(\alpha, \widehat{\theta}_T)$ the out-of-sample mean of $H_t(\alpha, \widehat{\theta}_T)$

$$\bar{H}(\alpha, \widehat{\theta}_T) = \frac{1}{n}\sum_{t=1}^{n} H_t(\alpha, \widehat{\theta}_T).$$

Estimation risk

1. Without estimation risk and when $n \longrightarrow \infty$, we have

$$UC_{MES}\left(\alpha, \theta_0\right) = \frac{\sqrt{n}\left(\bar{H}(\alpha, \theta_0) - \alpha/2\right)}{\sqrt{\alpha\left(1/3 - \alpha/4\right)}} \xrightarrow{d} \mathcal{N}\left(0, 1\right)$$

2. A similar result holds for the feasible statistic

$$UC_{MES} \equiv UC_{MES}(\alpha, \widehat{\theta}_T)$$

when $T \longrightarrow \infty$ and $n \longrightarrow \infty$, whereas $\lambda = n/T \longrightarrow 0$, i.e. when there is no estimation risk.

Estimation risk

$$UC_{MES} = \underbrace{\frac{1}{\sigma_H \sqrt{n}} \sum_{t=1}^{n} \left(H_t\left(\alpha, \theta_0\right) - \alpha/2\right)}_{UC_{MES}(\theta_0)}$$

$$+ \underbrace{\frac{\sqrt{\lambda}}{\sigma_H} \frac{1}{n} \sum_{t=1}^{n} \mathbb{E}\left(\left. \frac{\partial H_t\left(\alpha, \widetilde{\theta}\right)'}{\partial \theta} \right| \Omega_{t-1} \right) \sqrt{T}\left(\widehat{\theta}_T - \theta_0\right)}_{\text{Estimation risk}} + o_p\left(1\right.$$

Theorem (UC test statistic)

Under assumptions A1-A4, when $T \to \infty$, $n \to \infty$ *and* $n/T \to \lambda$ *with* $0 < \lambda < \infty$

$$UC_{MES} \xrightarrow{d} \mathcal{N}\left(0, \sigma_\lambda^2\right),$$

where the asymptotic variance σ_λ^2 *is*

$$\sigma_\lambda^2 = 1 + \lambda \frac{R'_{MES} \Sigma_0 R_{MES}}{\alpha \left(1/3 - \alpha/4\right)},$$

where $R_{MES} = \mathbb{E}_0 \left(\partial H_t \left(\alpha, \theta_0\right) / \partial \theta\right)$ *and* $\mathbb{V}_{as}(\hat{\theta}_T) = \Sigma_0 / T$.

Corollary (robust test statistic)

When $T \to \infty$, $n \to \infty$ *and* $n/T \to \lambda$ *with* $0 < \lambda < \infty$

$$UC^c_{MES} = \frac{\sqrt{n}\left(\bar{H}(\alpha, \widehat{\theta}_T) - \alpha/2\right)}{\left(\alpha\left(1/3 - \alpha/4\right) + n\widehat{R}'_{MES}\widehat{\mathbb{V}}_{as}(\hat{\theta}_T)\widehat{R}_{MES}\right)^{1/2}} \xrightarrow{d} \mathcal{N}(0,1)$$

with $\widehat{R}_{MES}$ *a consistent estimator of* R_{MES} *given by*

$$\widehat{R}_{MES} = -\frac{1}{\alpha n}\sum_{t=1}^{n}\frac{\partial F(\widehat{y}_t, \widehat{\theta}_T)}{\partial \theta} \times \mathbf{1}(y_{2t} \leq VaR_{2t}(\alpha, \hat{\theta}_T)).$$

and $\widehat{y}_t = (y_{1t}, VaR_{2t}(\alpha, \hat{\theta}_T))'$.

Independence test

The Box-Pierce test statistic for $H_{0,IND}$ is defined as

$$IND_{MES} = n \sum_{j=1}^{m} \hat{\rho}_j^2,$$

$$\hat{\rho}_j = \frac{\widehat{\gamma}_j}{\widehat{\gamma}_0},$$

$$\widehat{\gamma}_j = \frac{1}{n-j} \sum_{t=1+j}^{n} \left(H_t(\alpha, \widehat{\theta}_T) - \alpha/2 \right) \left(H_{t-j}(\alpha, \widehat{\theta}_T) - \alpha/2 \right),$$

Theorem (IND test statistic)

When $T \longrightarrow \infty$, $n \longrightarrow \infty$ *and* $n/T \longrightarrow \lambda$ *with* $0 < \lambda < \infty$

$$IND_{MES} \xrightarrow{d} \sum_{j=1}^{m} \pi_j Z_j^2,$$

where $\{\pi_j\}_{j=1}^{m}$ *are the eigenvalues of the matrix* Δ *with the ij-th element given by*

$$\Delta_{ij} = \delta_{ij} + \lambda R_i' \Sigma_0 R_j,$$

$$R_j = \frac{1}{\alpha\left(1/3 - \alpha/4\right)} \mathbb{E}_0 \left(\left(H_{t-j}(\alpha, \theta_0) - \alpha/2\right) \frac{\partial H_t(\alpha, \theta_0)}{\partial \theta} \right),$$

δ_{ij} *is a dummy variable that takes a value 1 if* $i = j$ *and 0 otherwise,* $\{Z_j\}_{j=1}^{m}$ *are independent standard normal variables.*

Monte Carlo simulation study

- Bivariate normal DGP

$$Y_t \overset{\text{i.i.d.}}{\sim} \mathcal{N}(\mathbf{0}, \Sigma)$$

$$\Sigma = \begin{pmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{pmatrix}$$

- Calibration: JP Morgan Chase and S&P500 index from January 1st, 2005 to October 9th, 2015.

		$UC_{MES}(\widehat{\theta}_T)$	$UC^C_{MES}(\widehat{\theta}_T)$	$IND_{MES}(\widehat{\theta}_T)$	$IND^C_{MES}(\widehat{\theta}_T)$
		$T=250, n=250$, Size and Power			
H_0	—	0.089	0.047	0.094	0.075
H_1^A	$\Delta\sigma_1^2 = 25\%$	0.374	0.332	0.073	0.064
	$\Delta\sigma_1^2 = 50\%$	0.882	0.871	0.109	0.087
	$\Delta\sigma_1^2 = 75\%$	0.997	0.997	0.264	0.215
H_1^B	$\Delta\sigma_2^2 = 25\%$	0.481	0.457	0.076	0.065
	$\Delta\sigma_2^2 = 50\%$	0.983	0.981	0.192	0.120
	$\Delta\sigma_2^2 = 75\%$	1.000	1.000	0.865	0.728
H_1^C	$\rho_{H_1} = 0.6$	0.444	0.396	0.080	0.059
	$\rho_{H_1} = 0.3$	0.760	0.750	0.092	0.068

		$UC_{MES}(\widehat{\theta_T})$	$UC^C_{MES}(\widehat{\theta_T})$	$IND_{MES}(\widehat{\theta_T})$	$IND^C_{MES}(\widehat{\theta_T})$
		$T = 250, n = 2500$ Size and Power			
H_0	—	0.312	0.045	0.076	0.056
H_1^A	$\Delta\sigma_1^2 = 25\%$	0.938	0.876	0.101	0.061
	$\Delta\sigma_1^2 = 50\%$	1.000	1.000	0.455	0.278
	$\Delta\sigma_1^2 = 75\%$	1.000	1.000	0.988	0.959
H_1^B	$\Delta\sigma_2^2 = 25\%$	0.997	0.994	0.119	0.082
	$\Delta\sigma_2^2 = 50\%$	1.000	1.000	0.937	0.785
	$\Delta\sigma_2^2 = 75\%$	1.000	1.000	1.000	1.000
H_1^C	$\rho_{H_1} = 0.6$	0.974	0.919	0.118	0.071
	$\rho_{H_1} = 0.3$	1.000	0.999	0.301	0.196

		$UC_{MES}(\widehat{\theta}_T)$	$UC^C_{MES}(\widehat{\theta}_T)$	$IND_{MES}(\widehat{\theta}_T)$	$IND^C_{MES}(\widehat{\theta}_T)$
		$T = 2500, n = 250$ Size and Power			
H_0	—	0.059	0.054	0.099	0.094
H_1^A	$\Delta\sigma_1^2 = 25\%$	0.359	0.357	0.075	0.074
	$\Delta\sigma_1^2 = 50\%$	0.884	0.884	0.113	0.109
	$\Delta\sigma_1^2 = 75\%$	0.999	0.999	0.242	0.240
H_1^B	$\Delta\sigma_2^2 = 25\%$	0.469	0.465	0.096	0.095
	$\Delta\sigma_2^2 = 50\%$	0.988	0.988	0.169	0.166
	$\Delta\sigma_2^2 = 75\%$	1.000	1.000	0.881	0.876
H_1^C	$\rho_{H_1} = 0.6$	0.417	0.413	0.082	0.082
	$\rho_{H_1} = 0.3$	0.756	0.754	0.080	0.077

Backtesting other Systemic Risk Measures

Other systemic risk measures can be backtested according to our methodology:

1. ΔCoVaR (Adrian and Brunnermeier 2016).
2. SES (Acharya et al. 2010).
3. SRISK (Acharya et al. 2012 and Brownlees and Engle 2015).

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Definition

Capital shortfall = capital reserves the firm needs to hold (regulation or prudential management) - the firm's equity.

$$CS_{1t-1} = k\left(L_{1t-1} + W_{1t-1}\right) - W_{1t-1}$$

As a consequence

$$\begin{aligned} SRISK_{1t} &= \mathbb{E}_{t-1}\left(CS_{1t} \middle| Y_{2t} < C \right) \\ &= k\mathbb{E}_{t-1}\left(L_{1t} \middle| Y_{2t} < C \right) - (1-k)\,\mathbb{E}_{t-1}\left(W_{1t} \middle| Y_{2t} < C \right) \end{aligned}$$

with L_{1t} the book value of debt, W_{1t} the market value of equity and k a prudential ratio.

Assumptions:

- $\mathbb{E}_{t-1}\left(L_{1t} \middle| Y_{2t} < C \right) = L_{1t-1}$.
- $\mathbb{E}_{t-1}\left(W_{1t} \middle| Y_{2t} < C \right) = W_{1t-1}(1 + \mathbb{E}_{t-1}\left(Y_{1t} \middle| Y_{2t} < C \right))$.

Definition (SRISK-MES)

Acharya et al. (2012) and Brownlees and Engle (2015) show that

$$SRISK_{1t} = k \; L_{1t-1} - (1-k) \, W_{1t-1} MES_{1t} \left(C \right)$$

Definition (Systemic risk measure)

Consider a conditioning event $C = VaR_{2t}\left(\alpha\right)$ and define RM_{it}, a MES-based systemic risk measure such that

$$RM_{1t}\left(\alpha, \theta_0\right) = g_t\left(MES_{1t}(\alpha, \theta_0), X_{t-1}\right)$$

with $g_t\left(.\right)$ an decreasing function (with MES) and X_{t-1} a set of variables that belong to Ω_{t-1}.

Backtesting

Define a violation process

$$\begin{aligned} & h_t(\alpha, \beta, \theta_0, X_{t-1}) \\ = \; & (1 - \mathbf{1}\left(g_t\left(Y_{1t}, X_{t-1}\right) \leq g_t\left(CoVaR_{1t}\left(\beta, \alpha, \theta_0\right), X_{t-1}\right)\right)) \times \\ & \times \mathbf{1}\left(Y_{2t} \leq VaR_{2t}\left(\alpha, \theta_0\right)\right) \end{aligned}$$

and a cumulative violation process

$$H_t\left(\alpha, \theta_0, X_{t-1}\right) = \int_0^1 h_t(\alpha, \beta, \theta_0, X_{it-1}) \mathrm{d}\beta.$$

Backtesting

We consider the following test

$$H_0 : \mathbb{E}\left(H_t\left(\alpha, \theta_0, X_{t-1}\right)\right) = \frac{\alpha}{2}.$$

The test statistic is then similar to that used for the MES test

$$CC_t = \frac{\sqrt{n}\left(\bar{H}(\alpha, \widehat{\theta}_T) - \alpha/2\right)}{\sqrt{\alpha\left(1/3 - \alpha/4\right)}},$$

with

$$\bar{H}(\alpha, \widehat{\theta}_T) = \frac{1}{n}\sum_{t=1}^{n} H_t(\alpha, \widehat{\theta}_T, X_{t-1}).$$

Conclusion

1. We propose two backtests for the UC and the IND hypothesis of the MES. These tests can be adapted in order to be robust to the presence of estimation risk.
2. The backtests are based on the concept of cumulative violation (Du and Escanciano, 2016) but they are similar to the traditional tests used for the VaR (Kupiec, 1995).
3. Empirical application: to be done...