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Testing for zeros in the spectrum of an univariate stationary process : Part I *

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ABSTRACT: Non parametric and parametric estimation for the spectral density of a stationary process is a well-known topic, except when the spectrum vanishes for some frequency. Indeed, for this frequency, the limit law degenerates, and traditional inference no longer applies. The paper introduces non parametric tests of this hypothesis, which exploit the asymptotic behavior of the periodogram for some well-chosen sequence of frequencies. In particular, statistics free from nuisance parameters are derived, and conditional heteroskedasticity of unknown form is allowed. As an application, stationarity tests against seasonal unit-root alternatives are developped.

RESUME: L'estimation de la densité spectrale d'un processus stationnaire univarié relève de procédures désormais classiques, sauf dans le cas où cette densité s'annule en un point. Pour une fréquence θ fixée, le papier introduit plusieurs tests non-paramétriques de l'hypothèse de nullité du spectre en θ fondé sur le périodogramme pris en un point voisin de θ. Ces tests se distinguent les uns des autres par la vitesse à laquelle cette fréquence converge vers θ. Nous dérivons ces résultats pour un processus linéaire avec hétéroscédasticité conditionnelle de forme inconnue. Un test de l'hypothèse de stationnarité contre l'alternative de racine unitaire saisonnière illustre ces résultats.

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1 Introduction

Let $(X_t)_{t\in\mathbb{Z}}$ be a stationary process with autocovariance function $\gamma(h)$ and spectral density $f(\omega) = \frac{1}{2\pi}\sum_{k=-\infty}^{+\infty}\gamma(k)e^{-ik\omega}$. We suppose a sample of size n is available to the econometrician. In the semi-parametric framework defined by the family of densities $f(\omega) \equiv f(\omega;\phi)$ where $\phi \in \Theta$ is a finite unknown parameter, it is well known (see Davies (1983) for a survey) that standard inference procedures are valid only if the following condition is satisfied "$\forall\,(\omega,\phi) \in [-\pi,\pi]\times\Theta,\ f(\omega;\phi) > 0$". In the time domain, maximum likelihood estimation for $ARMA$ models display "good" properties (asymptotic normality at rate $\sqrt{n}$) only if the model is invertible, which means that $f(\omega) > 0$ for all ω. When a root lies in the spectrum, classical estimators are therefore no longer asymptotically normal, and inference can't be carried out through standard arguments, like student statistics and so on. Note that the same trouble occurs for non parametric estimators of the spectral density which use prewithening of the data as a prior treatment. For such a procedure to be valid, the spectrum must be bounded away from above by a positive constant. Hence, it appears of primary importance to decide whether or not a root lies in the spectrum of the process before the application of any econometric procedure.

Such situations are not only of theoretical interest. Indeed, they may occur quite often in applied econometric works, when some variables are deliberately over differenced. A leading example of such misspecification is the use of annual differences of variables, which may imply in some cases for quarterly data, that the spectrum vanishes at both seasonal frequency $\frac{\pi}{2}$ and π. Another example is provided by seasonal adjustment procedure. More precisely, the widely used seasonal adjustment program, Census-X11, may be approximated as a linear filter, which can be factorized as $C(B)\,X(B)$[1] (Laroque (1977)), where $X(B)$ is a bilateral filter without roots on the unit circle, and:

$$C(B) = (1+B)^2\left(1+B^2\right)^2$$

In other words, when we apply this filter to a non stationary SARIMA process X_t such as:

$$(1-B)^{d_0}\,(1+B)^{d_\pi}\left(1+B^2\right)^{d_{\frac{\pi}{2}}} X_t = Z_t \qquad (1)$$

with $d_0, d_\pi, d_{\frac{\pi}{2}}$ integers and Z_t a stationary SARMA process with strictly positive spectrum at frequency $\omega \in \left\{0, \frac{\pi}{2}, \pi\right\}$, the seasonally adjusted series, Y_t, also admits a SARIMA representation, but its spectrum vanishes at $\frac{\pi}{2}$ if $d_{\frac{\pi}{2}} < 2$, and at π if $d_\pi < 2$. It is generally acknowledged that raw series built upon constant price indexes are generally integrated of order one or stationary, whereas data built upon current price indexes may be integrated of order two. In the former case, we may expect the seasonal adjusted series to be over-differenced, a phenomenon which may be also associated to the over seasonal correction of the procedure[2]. Moreover, it seems that this drawback is sometimes still present after individual seasonally adjusted series have been aggregated: with US data, Maravall (1995) found evidences that large lags

[1] B is the usual backshift operator.

[2] Model based methods which use data dependent Wiener-Kolmogorov filter may also induce moving average unit roots at seasonal frequencies (see Maravall (1995)).

are significant in autoregressive models. One practical consequence of such findings is that VAR models using such variables are misspecified.

The test of the null hypothesis $f(\theta) = 0$ appears to be non standard. Indeed, classical non-parametric estimators obtained by smoothing the periodogram are not useful, because of the form of their asymptotic variance: it is proportional to $f^2(\theta)$ (e.g. Priestley (1988)), thus the limit law is degenerate when $f(\theta) = 0$. Specific procedures must be developed. In the parametric framework, such tests are already available: thus, for the model

$$X_t = \varepsilon_t - \theta\varepsilon_{t-1}, \varepsilon_t \; iid \; \mathrm{N}\left(0, \sigma^2\right)$$

it is possible to perform the locally most powerful and invariant test of the hypothesis $\mathbf{H}_0 : \theta = 1$ (i.e. $f(0) = 0$) against $\mathbf{H}_a : |\theta| < 1$ (i.e. $f(0) > 0$) (see Tanaka (1996)). Further developments in this framework have been carried out by Tam and Reinsel (1997) in order to allow for seasonal moving average unit roots and autoregressive components.

For a given fixed frequency $\theta \in [0, \pi]$, this article develops a new test of the hypothesis $f(\theta) = 0$ which exploits the behavior of the periodogram for frequencies close (but not equal) to θ. Heteroskedasticity of the innovation of X_t is allowed. Such an approach is classical in the statistical analysis of long memory processes. In this framework and for $\theta = 0$, Lobato and Robinson (1998) have recently proposed a test of stationary hypotheses against fractionally alternatives. Their test uses values of the periodogram in a band of frequency centered around θ which degenerates asymptotically. Akdi and Dickey (1998) have also studied the behavior of $I\left(\frac{2k\pi}{n}\right)$ for fixed k and $n \to \infty$ when X_t is an ARIMA process ($I(.)$ is the periodogram). Our approach implies the use of only one frequency θ_n which converges to θ at a rate lower than n^{-1}. The intuition behind this approach is the following: *if* $f(\theta) = 0$, *then the periodogram is a consistent estimator of* $f(\theta)$ (see e.g. Brockwell and Davis (1986), theorem 10-3-2). Thus, we hope that this interesting property is still valid in an asymptotically degenerate neighborhood of θ. Moreover, the test is built upon local properties of the spectrum, thus it is possible to accommodate for irregularities in other areas of $[-\pi, \pi]$ e.g. divergence to infinity in case of long memory dynamic at some frequency $\widetilde{\theta} \neq \theta$: this topic will be studied in a subsequent paper.

As it is well-known, the problem under consideration is closely related to unit root tests. Indeed consider X_t defined by:

$$\left(1 - 2\cos\theta\lambda B + \lambda^2 B^2\right) X_t = \varepsilon_t$$

with ε_t stationary process with $f_\varepsilon(\theta) > 0$ and $0 \leq \lambda \leq 1$. Define:

$$Y_t = \left(1 - 2\cos\theta B + B^2\right) X_t$$

Let f_ε and f_Y be the spectral density of ε_t and Y_t respectively. We have:

If $\lambda = 1$ (unit-root hypothesis), $f_Y(\theta) > 0$
If $\lambda < 1$ (stationary alternative), $f_Y(\theta) = 0$

Note however, that the null and alternative hypotheses are reversed.

The paper is organized as follows. We begin with the possible representations of a stationary process with zeros in its spectrum. It is seen that the conclusions crucially depend on the local smoothness of the spectral density. Section 3 is devoted to a functional central limit theorem which provides the grounds for the first test statistics developed in section 4. Unfortunately, these statistics depend upon nuisance parameters which must be estimated. Section 5 develops an alternative approach which avoids this step. Section 6 discusses non-zero mean processes and then suggests a new test for the presence of seasonal unit roots. Lastly, we provide some elements for the case where we do not have prior knowledge of θ. Technical details are brought together in an appendix.

Now, we review the notations used throughout the paper.

We assume that a finite sample of data is available $\{X_k, 1 \leq k \leq n\}$, extracted from a purely non-deterministic stationary process (PND, in short) $(X_t)_{t\in\mathbb{Z}}$ with expectation zero (the case $E(X_t) \neq 0$ is deferred to the end of the paper). Its Wold representation takes the form:

$$X_t = \sum_{j=0}^{\infty} \Psi_j \varepsilon_{t-j} \tag{2}$$

$(\varepsilon_t)_{t\in\mathbb{Z}}$ is the innovation process of X_t, $\Psi_0 = 1$, $\sum_{j=0}^{\infty} \Psi_j^2 < \infty$. We define the complex function associated with the Wold expansion of X_t as $C_X(z) = \sum_{j=0}^{\infty} \Psi_j z^j$ for $|z| \leq 1$. The spectral density of X_t is then expressed as $f(\omega) = \frac{\sigma^2}{2\pi} \left|C_X\left(e^{-i\omega}\right)\right|^2$ with $\sigma^2 = \mathbb{E}\left(\varepsilon_t^2\right)$.

For any complex stationary process $(Z_t)_{t\in\mathbb{Z}}$, and $m \in \mathbb{Z}$, we denote by $H_Z(m) = \overline{\{Z_t, t \leq m\}}$ the complex Hilbert space spanned by $(Z_t)_{t\leq m}$, with $H_Z = H_Z(+\infty)$.

Let $\Delta_n(u) = \sum_{k=0}^{n-1} e^{iku}$ be the Dirichlet kernel,; it satisfies $\int_{-\pi}^{\pi} |\Delta_n(x)|^2 dx = n$ and:

$$|x\Delta_n(x)| \leq 2 \text{ for } 0 \leq |x| < \pi \tag{3}$$

The symbol "$\Rightarrow$" will always refer to convergence in law when n goes to infinity. $C[a,b]$ is the space of continuous functions defined on $[a,b]$, equipped with the supremum norm. $C, C', C'', \ldots$ are constant independent from integer or real variables n and $t \in [0,1]$, $\omega \in [0,\pi]$ used throughout the paper. These constants will take different values for each new occurrence. For $p \geq 1$, $\|X\|_p = \left(\mathbb{E}|X|^p\right)^{1/p}$ for any complex stochastic variable X with finite moments up to order p. Lastly, $[x]$ is the integer part of x.

2 Regularity and nullity of the spectrum

2.1 Local regularity

In the $ARMA$ framework, it is easy to see that when $\Phi(B) X_t = \Theta(B) \varepsilon_t$, and $f(\theta) = 0$, then $\Theta(B)$ can be factorized as: $\Theta(B) = \left(1 - 2\cos\theta B + B^2\right)^d \widetilde{\Theta}(B)$ d integer ≥ 1. Such a factorization is in fact possible under much more general conditions that we recall in the following result:

Lemma 1 *If* $\exists C, \alpha > 0$ *and* $V(\theta)$ *neighborhood of* θ *such as* $f(\omega) \leq C|\omega - \theta|^{1+\alpha}$ *for* $\omega \in V(\theta)$, *then there exists a unique real stationary process* u_t *such as:*
i) $X_t = (1 - 2\cos\theta B + B^2) u_t$ *if* $\theta \notin \{0, \pi\}$, $X_t = (1 - \cos\theta B) u_t$ *otherwise.*
ii) $\forall t \in \mathbb{Z}$, $u_t \in H_X(t)$

Proof: see the appendix .■

Note that the assumptions of this lemma imply the continuity and the derivability of f at θ with $f(\theta) = 0$ and (necessarily) $f^{'}(\theta) = 0$. When f is C^2 in a neighborhood of θ, the assumption $f(\theta) = 0$ implies automatically that the assumptions of the lemma are fulfilled with $\alpha = 1$.
We now define $\Delta(B, \theta) = 1 - 2\cos\theta B + B^2$ if $\theta \neq 0$ and $\theta \neq \pi$, $\Delta(B, \theta) = 1 - \cos\theta B$ otherwise.

Remark 1 *The unicity of* u_t *breaks down if we just impose the condition "*u_t *stationary". Indeed,* $\widetilde{u_t} = u_t + A\cos(\theta t) + B\sin(\theta t)$ *with* A *and* B *real random variables* $\in H_X^{\perp}$, *such as* $E(AB) = 0$, $E(A^2) = E(B^2) = \widetilde{\sigma}^2$, *is a solution.*

The spectral density of u_t takes the form (when $\theta \notin \{0, \pi\}$):
$f_u(\omega) = \frac{f(\omega)}{4\sin^2([\omega - \theta]/2)}$. If $f(\omega) \sim_{\omega \to \theta} C|\omega - \theta|^{1+\alpha}$, then $f_u(\omega) \sim_{\omega \to \theta} C|\omega - \theta|^{1+\widetilde{\alpha}}$ with $\widetilde{\alpha} = \alpha - 2$. f_u admits a singularity at θ if $\widetilde{\alpha} < 0$, and is continuous at θ if $\widetilde{\alpha} = 0$. When $\widetilde{\alpha} > 0$, one can apply the lemma 1 to u_t, and then we obtain by recurrence:

Corollary 2 *If* $\exists C > 0, \alpha > 1$ *such as* $f(\omega) \leq C|\omega - \theta|^{\alpha}$ *for* $\omega \in V(\theta)$, *then there exists one and only one stationary process* $u_t \in H_X(t)$ *such as, if* $d = -\left[\frac{1-\alpha}{2}\right]$:

$$X_t = \Delta(B, \theta)^d u_t$$

It is worth noting that a process may admit factorizations with indefinitely large d: consider for example a PND process X_t with spectral density $f(\omega) = \exp\left(-\frac{1}{\sqrt{|\omega - \theta|}}\right)$ for $\omega \in [0, \pi]$, and then extended to $\mathbb{R}$ by parity and 2π-periodicity (note that $\log f$ is integrable, as it should be). f is indefinitely differentiable at $\omega = \theta$, its derivatives all taking the value zero at this point. Hence, $\forall m \geq 1$, $\exists C_m > 0$ and $V_m(\theta)$ neighborhood of θ such as $f(\omega) \leq C_m|\omega - \theta|^m$ for all $\omega \in V_m(\theta)$. It follows that X_t can be written $X_t = (1 - 2\cos\theta B + B^2)^p u_t^{(p)}$ for each $p \geq 1$, with $u_t^{(p)}$ stationary.

The process u_t which appear in the course of lemma 1 can be explicited with the Wold expansion of X_t if we assume that f is globaly regular. This point is made precise in the following section.

2.2 Global regularity

Let $\mathbf{H}_d$ denote the following hypothesis:

$$\exists d \geq 0 \text{ such as} : \sum_0^{+\infty} j^d |\Psi_j| < \infty\text{"}$$

X_t is then said to be linearly regular, since when $d \in \mathbb{N}$, and $\mathbf{H}_d$ is satisfied, then f is C^d on the real line. It is in fact easy to give a reciprocal property

Lemma 3 *If f is C^p, $p \geq 2$, with $f^{(p)}$ piecewise C^1, and if f doesn't vanish, then f satisfies $\mathbf{H}_d$ for all $d < p-1$.*

As a direct consequence, we have:

Lemma 4 *Let f be C^p, $p \geq 2$, with $f^{(p)}$ piecewise C^1. Suppose that $f > 0$ except maybe on a finite set of values θ_k, $k = 1, \ldots, p$, where $f(\omega) \sim_{\omega \to \theta_k} C_k |\omega - \theta_k|^{2d_k}$, d_k integer ≥ 0. Then f satisfies $\mathbf{H}_d$ for $d < p-1$.*

Proof: From the corollary 2 applied to each frequency θ_k verifying $d_k \geq 1$, there exists a stationary process u_t such as, if we write $\widetilde{d_k} = d_k$ for $\theta_k \notin \{0, \pi\}$, and $\widetilde{d_k} = d_k/2$ for $\theta_k \in \{0, \pi\}$:
$X_t = \prod_{d_k \geq 1} (1 - 2\cos\theta_k B + B^2)^{\widetilde{d_k}} u_t$ with $f_u > 0$ and verifying the assumptions of lemma 3. Once applied to u_t, the conclusions of the lemma are preserved after application of the finite moving average polynomial $\prod_{1 \leq k \leq p, d_k \geq 1} (1 - 2\cos\theta_k B + B^2)^{\widetilde{d_k}}$

■

Now, we expand the function $C_X(z)$ around $C_X(e^{-i\theta})$. Indeed, for $p > 2$ and any θ, this lemma says that $\mathbf{H}_d$ is satisfied for some $d > 1$ and:

$$C_X(z) - C_X(e^{-i\theta}) = \sum_{j=0}^{\infty} \Psi_j \left(z^j - e^{-i\theta j}\right) = -\left(1 - ze^{i\theta}\right) C_X(\theta, z)$$

with $C_X(\theta, z) = \sum_{k=0}^{\infty} e^{ik\theta} \left(\sum_{j=k+1}^{\infty} \Psi_j e^{-ij\theta}\right) z^k$
$C_X(\theta, z)$ is continuous in the unit disk $|z| \leq 1$ since, from lemma 4 with $d \geq 1$

$$\sum_{k=0}^{\infty} \sum_{j=k+1}^{\infty} |\Psi_j| = \sum_{j=1}^{\infty} (j+1) |\Psi_j| < \infty$$

We denote by $a(k, \theta)$ the coefficients of the expansion $C_X(\theta, z)$ in power of z :

$$a(k, \theta) = e^{ik\theta} \left(\sum_{j=k+1}^{\infty} \Psi_j e^{-ij\theta}\right) \tag{4}$$

Under the hypotheses of lemma 4, suppose that $\theta = \theta_1$ with $d_1 \geq 1$. If $\theta \neq \pi$, the function $C_X(z)$ admits the conjugate root $e^{i\theta}$, and:

$$C_X(z) = -\left(1 - ze^{i\theta}\right) C_X(\theta, z) = \left(1 - ze^{i\theta}\right)\left(1 - ze^{-i\theta}\right) D(\theta, z)$$

The coefficients of $D(\theta, z)$ are then easily obtained:
$D(\theta, z) = \sum_{k=0}^{\infty} e^{-ik\theta} \left[\sum_{j=k+1}^{\infty} a(j, \theta) e^{ij\theta}\right] z^k = \sum_{k=0}^{\infty} b(k, \theta) z^k$, or:

$$b(k, \theta) = \frac{1}{\sin\theta} \operatorname{Im}\left[e^{i(k+1)\theta} \sum_{u=k+2}^{+\infty} \Psi_u e^{-i\theta u}\right] \tag{5}$$

$D(\theta, z)$ is continuous in the unit disk since $|b(k,\theta)| \leq C \sum_{u=k+2}^{\infty} |\Psi_u|$. If we suppose $p > 3$, then the preceding argument yields:

$$\sum_{k=1}^{\infty} k\,|b(k,\theta)| \leq C \sum_{k=1}^{\infty} \sum_{u=k+2}^{\infty} k\,|\Psi_u| \leq \sin(\theta) \sum_{u=0}^{\infty} \frac{(u-2)(u-1)}{2} |\Psi_u| < \infty$$

:and f_u satisfies $\mathbf{H}_d$ with $d = 1$.

We collect these results in:

Lemma 5 *Under the hypotheses of lemma (4) with $p > 2$, if $\theta = \theta_1 \notin \{0, \pi\}$ and $d_1 \geq 1$, X_t can be expressed as:*

$$X_t = \left(1 - 2\cos\theta + B^2\right) u_t \text{ with } u_t = D(\theta, B)\,\varepsilon_t$$

The Wold function of u_t is given by (5); f_u is C^p on $[-\pi, \pi] \setminus \{\theta\}$, and continuous at θ. If $p > 3$, f_u is C^1 in a neighborhood of θ.

Lastly, we remark that $C_X(\theta, z) = -\left(1 - ze^{-i\theta}\right) D(\theta, z)$ for all $z \neq e^{-i\theta}$, and then for $z = e^{i\theta}$ by continuity.
Hence, $C_X(\theta, e^{-i\theta}) = -\left(1 - e^{-i2\theta}\right) D(\theta, e^{-i\theta})$ and:

$$\sigma \left| C_X\left(\theta, e^{-i\theta}\right) \right| = 2\sin\theta \sqrt{2\pi f_u(\theta)} \tag{6}$$

When $\theta = \pi$, $X_t = (1 + B)\,u_t$, $f_u(\theta) = \frac{\sigma^2}{2\pi} \left| C_X\left(\theta, e^{-i\theta}\right) \right|^2$ and:

$$\sigma \left| C_X(\pi, -1) \right| = \sqrt{2\pi f_u(\pi)} \tag{7}$$

These results will be used later. From now on, we will work with the hypothesis $\mathbf{H}_d$ for d large enough. The preceding discussion showed that it is, in a certain sense equivalent to assume that f is smooth everywhere. This very strong assumption will permit us to easily derive results related to the the asymptotic properties of the periodogram of X_t in some neighborhood of θ.

3 A functional central limit theorem

3.1 Case 1: the innovation

We suppose throughout the paper that $(\varepsilon_t)_{t \in \mathbb{Z}}$ is a martingale difference sequence adapted to some filtration $(\mathbb{F}_t)$. The basic assumptions which we need to make are as follows:

$$\mathbf{H}_\varepsilon : \begin{array}{l} \mathbb{E}(\varepsilon_t^2 \mid \mathbb{F}_{t-1}) = \sigma_t^2,\ \mathbb{E}(\varepsilon_t^2) = \sigma^2 \\ \sigma_t^2 = \sigma^2 + \sum_{k=1}^{\infty} c_k \left(\varepsilon_{t-k}^2 - \sigma^2\right),\ \sum_{k=1}^{\infty} |c_k| < \infty \\ \sum_{k=1}^{\infty} c_k z^k \neq 1 \text{ if } |z| \leq 1 \\ \sup_t \mathbb{E}\left(\varepsilon_t^4\right) < \infty \end{array}$$

These hypotheses allow for ARCH or GARCH dynamic for ε_t, and are now common for the analysis of financial time series, such as interest rates or inflation which are typical applications we have in mind. Seo (1999) consider similar hypotheses for unit root tests with conditional heteroskedasticity. Note however that we do not suppose that the sequence $u_t = \frac{\varepsilon_t}{\sigma_t}$ is i.i.d. For example, consider the following GARCH(1,1) model:

$$\sigma_t^2 = \alpha_0 + \alpha_1 \varepsilon_{t-1}^2 + \beta_1 \sigma_{t-1}^2,\ \alpha_0 > 0, \alpha_1 \geq 0, \beta_1 \geq 0$$

If $\alpha_1 + \beta_1 < 1$, ε_t is square integrable, and:

$$\begin{aligned} \sigma_t^2 &= \alpha_0 (1-\beta_1)^{-1} + \alpha_1 \textstyle\sum_{k=1}^{\infty} \beta_1^{k-1} \varepsilon_{t-k}^2 \\ \sigma^2 &= \mathbb{E}(\varepsilon_t^2) = \alpha_0 (1-\alpha_1-\beta_1)^{-1} \end{aligned}$$

If moreover $3\alpha_1^2 + 2\alpha_1\beta_1 + \beta_1^2 < 1$, then moments up to order four of ε_t are finite (Bollerslev (1986)) and $\mathbb{E}(\varepsilon_t^4) = C^{ste}$. With a somewhat different control of heterogeneity of ε_t (*i.e* $\sigma_t^2 \equiv \sigma^2$, $\sup_t \mathbb{E}(\varepsilon_t^{2+\alpha} \mid \mathbb{F}_{t-1}) < \infty$ for some $\alpha > 0$), we have the following result (see Chan and Wei (1988)):

Theorem 6 *If $\theta \notin \{0,\pi\}$ and $S_n(t,\theta) = \sqrt{\frac{2}{n}} \left(\sum_{k=1}^{[nt]} \cos(k\theta)\varepsilon_k, \sum_{k=1}^{[nt]} \sin(k\theta)\varepsilon_k \right)'$ for $t \in [0,1]$, then the following convergence holds in $D[0,1]$, the space of real càdlàg functions defined on $[0,1]$ equipped with the Prokhorov metric:*

$$S_n(t,\theta) \Rightarrow \sigma \mathbb{W}_2(t)$$

$\mathbb{W}_2(t)$ is a standard 2-dimensional Brownian motion.
When $\theta \in \{0,\pi\}$, $S_n(t,\theta) = \sqrt{\frac{1}{n}} \left(\sum_{k=1}^{[nt]} \cos(k\theta)\varepsilon_k \right) \Rightarrow \sigma \mathbb{W}_r(t)$, $\mathbb{W}_r(t)$ standard real Brownian motion.

It is interesting to put this result in complex notation. When $\theta \notin \{0,\pi\}$:

$$\sqrt{\frac{2}{n}} \sum_{k=1}^{[nt]} e^{-ik\theta} \varepsilon_k \Rightarrow \sigma \mathbb{W}_c(t) \tag{8}$$

$\mathbb{W}_c(t)$ is a standard complex Brownian motion: $\mathrm{Re}(\mathbb{W}_c(t))$ and $\mathrm{Im}(\mathbb{W}_c(t))$ are two independent standard Brownian motions. Now, we extend this result for a sequence θ_n converging to θ. For the sake of simplicity, we will express our result in the space $C[0,1]$: this is the reason why additional terms appear in the statistics introduced below; of course, all these terms vanish asymptotically.

Remark 2 *For two frequencies θ_1 and $\theta_2 \in]0,\pi[$, $\theta_1 \neq \theta_2$, we have the stronger result:*

$$\sqrt{\frac{2}{n}} \begin{pmatrix} \sum_{k=1}^{[nt]} e^{-ik\theta_1} \varepsilon_k \\ \sum_{k=1}^{[nt]} e^{-ik\theta_2} \varepsilon_k \end{pmatrix} \Rightarrow \sigma \begin{pmatrix} \mathbb{W}_c^1(t) \\ \mathbb{W}_c^2(t) \end{pmatrix} \tag{9}$$

The two Brownian motions $\mathbb{W}_c^1$ are $\mathbb{W}_c^2$ independent.

The main tool used in the paper is:

Theorem 7 *Let $\theta_n \in \left]0,\pi\right[$ be a sequence converging to θ, and such as $n(\theta_n-\theta) \to -\infty$ if $\theta=\pi$, $n\theta_n \to +\infty$ if $\theta=0$. We define*

$$S_n(t,\theta_n)=\sqrt{\frac{2}{n}}\left(\sum_{k=1}^{[nt]}\cos(k\theta_n)\varepsilon_k,\sum_{k=1}^{[nt]}\sin(k\theta_n)\varepsilon_k\right)'$$

Let $T_n(t,\theta_n)$ denote the continuous function defined on $[0,1]$, by linear interpolation of $[S_n(k/n\ ,\theta_n)]_{k=0,\dots,n}$. Then under $\mathbf{H}_\varepsilon$, $T_n(t,\theta_n) \underset{n\to\infty}{\Rightarrow} \sigma\mathbb{W}(t), \mathbb{W}(t)$ standard 2-dimensional Brownian motion.

Proof: see the appendix.■

Note that this result do not suppose that θ_n converges rapidly to θ,and that the limiting distribution is the same as in the case $\theta_n=\theta$ for all n. In complex notation, this result takes the form:

$$\sqrt{\frac{2}{n}}\sum_{k=1}^{[nt]}e^{-ik\theta_n}\varepsilon_k \Rightarrow \sigma\mathbb{W}_c(t) \tag{10}$$

We omit the asymptotically negligible term $\sqrt{\frac{2}{n}}\left(e^{-ik\theta_{[nt]+1}}\varepsilon_{[nt]+1}\right)(nt-[nt])$, as we will do for the presentation of similar results in the sequel.

The case $\theta=\pi$ being quite similar to $\theta=0$, we suppose from now on, that $\theta\in\left]0,\pi\right]$.

3.2 Case 2: X_t

Let $\mathbf{H}$ denote the joined hypotheses $[\mathbf{H}_\varepsilon,\mathbf{H}_d,d\geq 1]$. We now introduce the finite Fourier transform of the observations $(X_t)_{1\leq k\leq n}$:

$$\mathbb{J}_X(\omega,t)=\frac{1}{\sqrt{n}}\sum_{k=1}^{[nt]}e^{-ik\omega}X_k,\text{ for } t\in[0,1] \text{ and } \omega\in[0,\pi]$$

We omit the dependance upon n of $\mathbb{J}_X(\omega,t)$ in order to shorten the notations. We also abbreviate $\mathbb{J}_X(\omega)\equiv\mathbb{J}_X(\omega,1)$ and $\mathbb{I}_X(\omega,t)=|\mathbb{J}_X(\omega,t)|^2$ where $\mathbb{I}_X(\omega)$ is the standard periodogram, $\mathbb{I}_X(\omega)=|\mathbb{J}_X(\omega)|^2$.
Following Phillips and Solo (1992), we have:
$X_t=C_X(B)\varepsilon_t=C_X(e^{-i\omega})\varepsilon_t-(1-e^{i\omega}B)\,C_X(\omega,B)\varepsilon_t$ thus:

$$\begin{aligned}\mathbb{J}_X(\omega,t)&=C_X(e^{-i\omega})\mathbb{J}_\varepsilon(\omega,t)-\tfrac{1}{\sqrt{n}}\textstyle\sum_{k=1}^{[nt]}(1-e^{i\omega}B)\,C_X(\omega,B)e^{-i\omega k}\varepsilon_t\\&=C_X(e^{-i\omega})\mathbb{J}_\varepsilon(\omega,t)-\tfrac{1}{\sqrt{n}}\left[e^{-i\omega[nt]}C_X(\omega,B)\varepsilon_{[nt]}-C_X(\omega,B)\varepsilon_0\right]\\&=C_X(e^{-i\omega})\mathbb{J}_\varepsilon(\omega,t)+R_n(\omega,t)\end{aligned}$$

with $R_n(\omega,t)=-\frac{1}{\sqrt{n}}\left[e^{-i\omega[nt]}C_X(\omega,B)\varepsilon_{[nt]}-C_X(\omega,B)\varepsilon_0\right]$
Define $\varepsilon(\omega,t)=C_X(\omega,B)\varepsilon_{[nt]}=\sum_{j=1}^{\infty}\Psi_j e^{-ij\omega}\left(\sum_{k=0}^{j-1}e^{ik\omega}\varepsilon_{[nt]-k}\right)\overset{def}{=}\sum_{j=1}^{\infty}u_{j,t,n}$

Since $\|u_{j,t,n}\|_2^2 = j\,|\Psi_j|^2\,\sigma^2$, $\|\varepsilon(\omega,t)\|_2^2 \leq \sigma \sum_{j=1}^{\infty} \sqrt{j}\,|\Psi_j| < \infty$, we obtain: $\mathbb{E}\left(|\varepsilon(\omega,t)|^2\right) = O\left(\frac{1}{n}\right)$ uniformly in ω and t. The second term in $R_n(\omega,t)$ verifies the same property, then: $\mathbb{E}\left(|R_n(\omega,t)|^2\right) = O\left(\frac{1}{n}\right)$ uniformly in ω and t. It yields that:

$$\mathbb{E}\left|\mathbb{I}_X(\theta_n,t) - \frac{2\pi}{\sigma^2}\mathbb{I}_\varepsilon(\theta_n,t)\,f(\theta_n)\right| = O\left(\frac{1}{\sqrt{n}}\right) \text{ uniformly in } t \tag{11}$$

For $t = 1$, this is the classical result which can be found in Priestley (1988). Moreover, unless (ε_t) is a white noise, the bound $O\left(n^{-1/2}\right)$ is the best possible in (11).

We now adopt the notation $\theta_n = \theta + e^{-1}(n)$ with $|e(n)| \to +\infty$

Theorem 8 *Under* **H***:*
i) If $f(\theta) = 0$ and $e(n) = o\left(n^{1/2}\right)$ then, for each $t_0 \in]0,1]$:

$$e(n)\mathbb{J}_X(\theta_n,t_0) \Rightarrow \frac{i\sigma C\left(\theta, e^{-i\theta}\right)}{\sqrt{2}}\mathbb{W}_c(t_0) \tag{12}$$

If moreover $e(n) = o\left(n^{1/4}\right)$ then

$$e(n)\mathbb{J}_X(\theta_n,t) \Rightarrow \frac{i\sigma C\left(\theta, e^{-i\theta}\right)}{\sqrt{2}}\mathbb{W}_c(t) \text{ in } C[0,1] \tag{13}$$

ii) If $f(\theta) \neq 0$ and $e(n) = o(n)$ for $\theta = \pi$:

$$\mathbb{J}_X(\theta_n,t) \Rightarrow \frac{\sigma C\left(e^{-i\theta}\right)}{\sqrt{2}}\mathbb{W}_c(t) \text{ in } C[0,1] \tag{14}$$

Proof:
If $f(\theta) = 0$, then $\forall x \in [0,\pi]$, $C\left(e^{-ix}\right) = -\left(1 - e^{i(\theta - x)}\right) C_X(\theta, e^{-ix})$ thus:

$$-e(n)\,\mathbb{J}_X(\theta_n,t) = -\left(\frac{1 - e^{i(\theta-\theta_n)}}{\theta - \theta_n}\right) C_X(\theta, e^{-i\theta_n})\mathbb{J}_\varepsilon(\theta_n,t) + \frac{R_n(\theta_n,t)}{\theta - \theta_n}$$

Since $\theta_n \to \theta$, $\frac{1-e^{i(\theta-\theta_n)}}{\theta-\theta_n} \to -i$ and $C_X(\theta, e^{-i\theta_n}) \to C_X(\theta, e^{-i\theta})$ by continuity. Lastly $\mathbb{E}\left(\left|\frac{R_n(\theta_n,t)}{\theta-\theta_n}\right|^2\right) \leq \frac{C}{n(\theta-\theta_n)^2}$, so if $\theta - \theta_n = o\left(\frac{1}{\sqrt{n}}\right)$, $\frac{R_n(\theta_n,t)}{\theta-\theta_n} \xrightarrow{P} 0$ uniformly in t, and the first part of statement $i)$ is proved.

We now want to obtain a stronger result. More precisely, we claim that:

$$\sup_{t\in[0,1]}\left|\frac{R_n(\theta_n,t)}{\theta - \theta_n}\right| \xrightarrow{P} 0 \tag{15}$$

Since $Y_n = \sup_{t\in[0,1]}\left|\frac{R_n(\theta_n,t)}{\theta-\theta_n}\right| \leq \max_{1\leq k\leq n} \frac{2}{\sqrt{n}}\left|\frac{C_X(\theta_n,B)\varepsilon_k}{\theta-\theta_n}\right|$ classical manipulations yields:

$$\begin{aligned} P(Y_n > 2\varepsilon) &\leq P\left(\textstyle\sum_{k=1}^n |C_X(\theta_n,B)\varepsilon_k| > \frac{\varepsilon\sqrt{n}}{|e(n)|}\right) \\ &\leq P\left(\textstyle\sum_{k=1}^n |C_X(\theta_n,B)\varepsilon_k|^2\,\mathbb{I}\left[|C_X(\theta_n,B)\varepsilon_k|^2 > \frac{\varepsilon^2 n}{e^2(n)}\right] > \frac{\varepsilon^2 n}{e^2(n)}\right) \\ &\leq P\left(\mathbb{J}_n > \varepsilon^2\right) \end{aligned}$$

with $\mathbb{J}_n = \frac{e^2(n)}{n}\sum_{k=1}^{n}|C_X(\theta_n,B)\varepsilon_k|^2\,\mathbb{I}\left[|C_X(\theta_n,B)\varepsilon_k|^2 > \frac{\varepsilon^2 n}{e^2(n)}\right]$.
In view of the proof of theorem 2, we only have to prove that $\mathbb{E}(\mathbb{J}_n) \to 0$.

$$\mathbb{E}\left(\mathbb{J}_n\right) \leq \frac{e^2(n)}{n}\sum_{k=1}^{n}\sqrt{\mathbb{E}\left(|C_X(\theta_n,B)\varepsilon_k|^4\right)}\sqrt{P\left(|C_X(\theta_n,B)\varepsilon_k|^2 > \frac{\varepsilon^2 n}{e^2(n)}\right)}$$

$$\begin{aligned}\|C_X(\theta_n,B)\varepsilon_k\|_4 &\leq \textstyle\sum_{j=0}^{\infty}|a(j,\theta_n)|\,\|\varepsilon_{k-j}\|_4\\ &\leq C\textstyle\sum_{j=0}^{\infty}|a(j,\theta_n)| \leq C\sum_{j=0}^{\infty}\left(\sum_{u=j+1}^{\infty}|\Psi_u|\right) < \infty\end{aligned}$$

Hence $\mathbb{E}\left(\mathbb{J}_n\right) \leq C\frac{e^2(n)}{n}\sum_{k=1}^{n}\sqrt{\left[\left(\frac{\varepsilon^2 n}{e^2(n)}\right)^{-2}\mathbb{E}\left(|C_X(\theta_n,B)\varepsilon_k|^4\right)\right]}$
and $\mathbb{E}\left(\mathbb{J}_n\right) \leq C\frac{e^4(n)}{\varepsilon^2 n} = o\left(1\right)$ since $\|C_X(\theta_n,B)\varepsilon_k\|_4$ is bounded, and (15) follows.

It is then easily seen that the continuous function $R_n^*(\theta_n,t)$ obtained by the linear interpolation of $R_n(\theta_n,\frac{k}{n})_{k=0,\ldots,n}$ also satisfies $\sup_{t\in[0,1]}\left|\frac{R_n^*(\theta_n,t)}{\theta-\theta_n}\right| \xrightarrow{P} 0$.

Since $\sqrt{2}\mathbb{J}_\varepsilon\left(\theta_n,t\right) + \sqrt{\frac{2}{n}}\left(e^{-ik\theta_{[nt]+1}}\varepsilon_{[nt]+1}\right)\left(nt-[nt]\right) \Rightarrow \sigma\mathbb{W}_c(t)$, we finally obtain:

$$\frac{\mathbb{J}_X\left(\theta_n,t\right)}{\theta-\theta_n} + a_n\left(t\right) \Rightarrow -\frac{i\sigma C\left(\theta,e^{-i\theta}\right)}{\sqrt{2}}\mathbb{W}_c(t)$$

where $a_n\left(t\right) = \frac{1}{\sqrt{n}(\theta-\theta_n)}\left(e^{-ik\theta_{[nt]+1}}\varepsilon_{[nt]+1}\right)\left(nt-[nt]\right) + \frac{R_n^*(\theta_n,t)-R_n(\theta_n,t)}{\sqrt{2}(\theta-\theta_n)}$
and (13) immediately follows, since $\sup_{t\in[0,1]}|a_n\left(t\right)| = o_p\left(1\right)$.

Now suppose that $f\left(\theta\right) \neq 0$. Then $\mathbb{J}_X\left(\theta_n,t\right) = C\left(e^{-i\theta_n}\right)\mathbb{J}_\varepsilon\left(\theta_n,t\right) + R_n\left(\theta_n,t\right)$ with $\sup_{t\in[0,1]}|R_n\left(\theta_n,t\right)| \xrightarrow{P} 0$. Thus, with the only requirement that $e(n) = o\left(n\right)$ when $\theta = \pi$, we get (14).

■

Remark 3 *When $\theta_n \equiv \theta$ for all n, we have, as a simple application of theorem 7 ;*

$$\mathbb{J}_X\left(\theta,t\right) \Rightarrow \frac{\sigma C\left(e^{-i\theta}\right)}{\sqrt{2}}\mathbb{W}_c(t)\ \textit{if}\ \theta \neq \pi,\ \ \sigma C\left(e^{-i\theta}\right)\mathbb{W}_r(t)\ \textit{if}\ \theta = \pi \qquad (16)$$

In the results (13), (14) and (16), it will be convenient to say that $\mathbb{W}_c$ is the Brownian motion associated to X_t

From these results, we can now deduce our statistics of interest. Two applications of the continuous mapping theorem yield, when $f(\theta) = 0$:

- If $e(n) = o\left(n^{1/4}\right)$:
  $\sup_{t\in[0,1]} e(n)\mathbb{J}_X\left(\theta_n,t\right) \Rightarrow \sigma\left|C_X\left(\theta,e^{-i\theta}\right)\right|/\sqrt{2}\times\sup_{t\in[0,1]}|\mathbb{W}_c(t)|$
  $$e^2(n)\sup_{t\in[0,1]}\mathbb{I}_X\left(\theta_n,t\right) \Rightarrow \sigma^2\left|C_X\left(\theta,e^{-i\theta}\right)\right|^2\Big/2\times\sup_{t\in[0,1]}|\mathbb{W}_c(t)|^2 \qquad (17)$$

  Note that $|\mathbb{W}_c(t)|$ is a Bessel process of order 2.

- If $e(n) = o\left(n^{1/2}\right)$:
  $\left|e(n)\right| \mathbb{J}_X\left(\theta_n, 1\right) \Rightarrow i\sigma C\left(\theta, e^{-i\theta}\right) \mathbb{W}_c(1)$, and:

$$e^2(n)\mathbb{I}_X\left(\theta_n\right) \Rightarrow \sigma^2 \left|C_X\left(\theta, e^{-i\theta}\right)\right|^2 \chi^2\left(2\right)/\, 2 \tag{18}$$

4 Test of the hypothesis: $f(\theta) = 0$

The hypothesis **H** is supposed to be satisfied, with $d \geq 2$.

4.1 Definition of the test statistics

We, first, must be more specific about the hypotheses to be tested. Indeed, when $f_u(\theta) = 0$, the limit laws which appear in (17) and (18) are degenerate, and we are faced once again with the problem raised in the introduction. Hence, we define the following null hypothesis:

$$\mathbf{H}_0 : f(\theta) = 0, f^{(2)}(\theta) \neq 0 \tag{19}$$

Note that $\mathbf{H}_0$ can also be written as: $f(\theta) = 0, f_u(\theta) \neq 0$.
The statistics defined by (17) and (18) depend on the nuisance parameter $\sigma C\left(\theta, e^{-i\theta}\right)$. From (6) and (7), this parameter depends only on $f_u\left(\theta\right)$. If the u_t could be observable, the tests statistics might be defined as:

$$\xi_n^s = \frac{e^2(n) \sup\limits_{0 \leq k \leq n} \mathbb{I}_X\left(\theta_n, k\,/n\right)}{\widehat{f_u(\theta)}}, \xi_n^p = \frac{e^2(n)\mathbb{I}_X\left(\theta_n\right)}{\widehat{f_u(\theta)}} \tag{20}$$

$\widehat{f_u(\theta)}$ is a consistent estimator of $f_u(\theta)$. The limit laws of these statistics are $K\left(\theta\right) \sup\limits_{t\in[0,1]} \left|\mathbb{W}_c(t)\right|^2$ and $K\left(\theta\right)\chi^2\left(2\right)$ respectively, with $K\left(\theta\right) = 4\pi\sin^2\theta$ if $\theta \neq \pi$, $K\left(\pi\right) = \pi$ if $\theta = \pi$. Note that these results are valid once $e(n) = o\left(n^{1/2}\right)$ for ξ_n^p, and $e(n) = o\left(n^{1/4}\right)$ for ξ_n^s.
Unfortunately, u_t can't be exactly recovered from X_t. Indeed, following Gregoir (1999), we define for any temporal series (Z_t), the series of cumulated sums at frequency ω $\left[S_t\left(Z,\omega\right)\right]_{t\in\mathbb{N}}$ in the following way

$$\begin{array}{l} S_t\left(Z,\omega\right) = \sum_{k=1}^t Z_k \frac{\sin(t+1-k)\omega}{\sin\omega} \text{ if } t > 0 \text{ and } \omega \neq \pi \\ S_t\left(Z,\pi\right) = \sum_{k=1}^t Z_k e^{-i\pi(t-k)} \text{ if } t > 0 \\ S_t\left(Z,\omega\right) = 0 \text{ if } t = 0 \end{array}$$

It is easily seen that:

$$\begin{array}{l} \left(1 - 2\cos\theta B + B^2\right) S_t\left(X,\theta\right) = X_t \\ S_t\left(X,\theta\right) = u_t - u_0 \frac{\sin\theta(t+1)}{\sin\theta} + u_{-1}\frac{\sin\theta t}{\sin\theta} \text{ if } \omega \neq \pi \\ S_t\left(X,\pi\right) = u_t - u_0\cos\left(\pi t\right) \end{array} \tag{21}$$

$S_t(X,\theta)$ includes some deterministic components[3] with stochastic oscillations around frequency θ. Thus, estimation of $f_u(\theta)$ from $S_t(X,\theta)$ with usual methods is not satisfactory in that case.

As in the case where u_0 and u_{-1} are constant, we propose to perform, for $\theta \neq \pi$, a regression of $S_t(X,\theta)$ on $\sin\theta(t+1)$ and $\sin\theta t$ for $t=1,\dots n$, and then collect the residuals $\widetilde{u}_{t,n}$ of this regression. For $\theta=\pi$, there is only one regressor: $\cos\pi t$.
Let $\mathbb{J}_{\widetilde{u}}(\omega,1)$ be the Fourier transform of the sequence $(\widetilde{u}_{t,n})_{1\leq t\leq n}$. It is clear that $\mathbb{J}_{\widetilde{u}}(\theta,t)=0$, and, for $\omega_j=\theta+\frac{2\pi j}{n}$, $j=-m,\dots,-1,1,\dots m$, and $m=o(n)$:

Lemma 9 $\left\|\mathbb{J}_{\widetilde{u}}(\omega_j,t)-\mathbb{J}_u(\omega_j,t)\right\|_2=O\left(\frac{1}{n}\right)$ *uniformly in* t.

Proof: we assume that $\theta\neq\pi$, the proof being identical for $\theta=\pi$. For ease of exposition, we write de regression model in its complex form:

$$S_t(X,\theta)=u_t+ae^{it\theta}+\overline{a}e^{-it\theta}$$

with $a\in H_X$. The matrix representation of this model is written as:

$$S_n=X_n\beta+U_n$$

Let $\beta=(a,\overline{a})$ and $\beta_n=(a_n,\overline{a}_n)$ the OLS estimator. $\widetilde{U}_n=(\widetilde{u}_{1,n},\dots,\widetilde{u}_{n,n})'=U_n-X_n\left(\overline{X}_n'X_n\right)^{-1}\overline{X}_n'U_n$
$\mathbb{J}_{\widetilde{u}}(\omega,t)-\mathbb{J}_u(\omega,t)=\frac{1}{\sqrt{n}}\sum_{h=1}^{[nt]}e^{-ih\omega}\left[\widetilde{u}_{h,n}-u_h\right]$, and some easy calculations yield:

$$\begin{aligned}\mathbb{J}_{\widetilde{u}}(\omega,t)-\mathbb{J}_u(\omega,t)= & -\frac{1}{n}\left(1-\frac{|\Delta_n(2\theta)|^2}{n^2}\right)^{-1}\times\left(\Delta_n(\theta-\omega)\left\{\mathbb{J}_u(\theta,t)-\frac{\Delta_n(-2\theta)}{n}\overline{\mathbb{J}_u(\theta,t)}\right\}\right.\\ & \left.+\Delta_n(-\theta-\omega)\left\{\overline{\mathbb{J}_u(\theta,t)}-\frac{\Delta_n(2\theta)}{n}\mathbb{J}_u(\theta,t)\right\}\right)\end{aligned}$$

But $\left\|\mathbb{J}_u(\theta,t)\right\|_2=O(1)$ uniformly in t since u_t is PND. Hence:

$$\left\|\mathbb{J}_{\widetilde{u}}(\omega,t)-\mathbb{J}_u(\omega,t)\right\|_2\leq\frac{1}{n}\left(C_1\left|\Delta_n(\theta-\omega)\right|+C_2\left|\Delta_n(\theta+\omega)\right|\right)$$

Since $\Delta_n(\theta-\omega_j)=0$ for $j\neq0$ and $|\theta+\omega_j|\leq2\theta+\frac{2\pi m}{n}<\pi$ for n large enough ($\theta\in]0,\pi[$ by assumption), the result follows from (3).
■

Remark 4 *If we use* $\mathbb{J}_{SX}(\omega,t)$ *instead of* $\mathbb{J}_{\widetilde{u}}(\omega_j,t)$, *we have the less precise bound:* $\left\|\mathbb{J}_{SX}(\omega_j,t)-\mathbb{J}_u(\omega_j,t)\right\|_2=O\left(\frac{1}{\sqrt{n}}\right)$ *uniformly in* t.

4.2 Convergence of the test

In this part we study the behavior of both statistics ξ_n^p and ξ_n^s under two distinct alternative schemes:

$$\begin{aligned}\mathbf{H}_a &: f(\theta)\neq0\\ \mathbf{H}_a' &: f(\theta)=0,f_u(\theta)=0\end{aligned}\qquad(22)$$

θ is a root of order two of f under $\mathbf{H}_0$, and of order four under $\mathbf{H}_a'$.

[3] in the precise sense of the Wold decomposition.

4.2.1 The periodogram under $\mathbf{H}_a$

First, suppose that $\theta \neq \pi$. Under $\mathbf{H}_a$ let $S_t(X,\theta) = Y_t$ and let $\widetilde{Y_{t,n}}$ denote the process obtained after the regression detailed in the previous part. For each $t \geq 2$:

$$\left(1 - 2\cos\theta B + B^2\right) Y_t = X_t$$

and Y_t is clearly not (asymptotically) stationary, since the autoregressive component admits the pair of complex unit roots $\left(e^{i\theta}, e^{-i\theta}\right)$. We now recall in a lemma the asymptotic behavior of $\mathbb{I}_Y(\omega)$.

Lemma 10 *Under $\mathbf{H}_a$, and j fixed integer:*
i) $\frac{\mathbb{J}_Y(\omega)}{n} = o_p(1)$ *if* $\omega \neq \theta$
ii) $\frac{1}{n}\mathbb{J}_Y\left(\theta + \frac{2\pi j}{n}\right) \Rightarrow -\frac{i\sigma e^{i\theta}}{2\sqrt{2}\sin\theta} C_X\left(e^{-i\theta}\right)\int_0^1 e^{-2i\pi jt}\mathbb{W}_c(t)dt$ *if* $\theta \neq \pi$
$\Rightarrow \sigma C_X\left(e^{-i\pi}\right)\int_0^1 e^{-2i\pi jt}\mathbb{W}_r(t)dt$ *if* $\theta = \pi$

Proof: see the appendix .■
We deduce:

$$\begin{aligned}\mathbb{J}_{\tilde{Y}}(\omega) &= \mathbb{J}_Y(\omega) - \tfrac{1}{n}\left(1 - \tfrac{|\Delta_n(2\theta)|^2}{n^2}\right)^{-1} \times \left(\Delta_n(\theta-\omega)\left\{\mathbb{J}_Y(\theta) - \tfrac{\Delta_n(-2\theta)}{n}\overline{\mathbb{J}_Y(\theta)}\right\}\right. \\ &\left.+ \Delta_n(-\theta-\omega)\left\{\overline{\mathbb{J}_Y(\theta)} - \tfrac{\Delta_n(2\theta)}{n}\mathbb{J}_Y(\theta)\right\}\right) = o_p(n) \text{ if } \omega \neq \theta\end{aligned}$$

For $\omega = \omega_j = \theta + \frac{2\pi j}{n}$ and $j \neq 0$, we get, since $\Delta_n\left(-\frac{2\pi j}{n}\right) = 0$:

$$\begin{aligned}\mathbb{J}_{\tilde{Y}}(\omega_j) &= \mathbb{J}_Y(\omega_j) + \tfrac{1}{n}(1 + o(1)) \times \Delta_n\left(-2\theta - \tfrac{2\pi j}{n}\right)\left\{\overline{\mathbb{J}_Y(\theta)} + O_p(1)\right\} \\ &= \mathbb{J}_Y(\omega_j) + O_p(1)\end{aligned}$$

The term $O_p(1)$ is uniform in j. It means that $\frac{1}{n}\mathbb{J}_{\tilde{Y}}(\omega_j)$ and $\frac{1}{n}\mathbb{J}_Y(\omega_j)$ have the same limit. On the other hand, the fact that $\mathbb{J}_{\tilde{Y}}(\theta) = 0$ is not very attractive under $\mathbf{H}_a$, because we want to obtain the divergence to infinity of some spectral estimator $\widehat{f_Y(\theta)}$. This is the reason why we introduce the following quantity:

$$\begin{aligned}\mathbb{J}^*_Y(\omega,t) &= \mathbb{J}_{\tilde{Y}}(\omega,t) \text{ if } \omega \neq \theta \\ &= \tfrac{\mathbb{J}_Y(\omega,t)}{n^\alpha} \text{ if } \omega = \theta,\ \alpha \in]0,1[\\ \text{and } \mathbb{I}^*_Y(\omega) &= \left|\mathbb{J}^*_Y(\omega)\right|^2\end{aligned}$$

We expect this slightly minor correction to improve finite sample performance of the test under the alternative $\mathbf{H}_a$, by making the divergence to infinity of $\widehat{f_Y(\theta)}$ faster.

4.2.2 A spectral density estimator

A general class of spectral density estimator are Daniell's estimators, that is, for $\theta \in [0,\pi]$ (see e.g. Brilinger (1981)):

$$\begin{aligned}\widehat{f(\theta)} &= \tfrac{1}{2\pi}\textstyle\sum_{j=-m}^m W_{m,j}\mathbb{I}\left(\widehat{\theta} + \theta_j\right) \text{ if } \theta \notin \{0,\pi\} \\ \widehat{f(\theta)} &= \tfrac{1}{2\pi}\left[\textstyle\sum_{j=1}^m W_{m,j}\mathbb{I}(\theta+\theta_j)\right] \Big/ \left[\textstyle\sum_{j=1}^m W_{m,j}\right] \text{ if } \theta \in \{0,\pi\}\end{aligned}$$

with $\widehat{\theta} = \frac{2\pi s(n)}{n}$, $\theta_j = \frac{2\pi j}{n}$, $s(n) = \left[\frac{n\theta}{2\pi}\right]$, $m = o(n)$; the positive weights $W_{m,j}$ satisfy the constraints $W_{m,j} = W_n(-j)$, $\sum_{j=-m}^{m} W_{m,j} = 1$ and $\sum_{j=-m}^{m} W_{m,j}^2 \to 0$ when $n \uparrow \infty$. The asymptotic behavior of $\mathbb{I}\left(\widehat{\theta} + \theta_j\right)$ is much easier to handle under $\mathbf{H}_a$, so we use a slightly modified estimator, with uniform weights:

$$\widehat{f^*(\theta)} = \frac{1}{2\pi} \times \frac{1}{2m+1} \sum_{j=-m}^{m} \mathbb{I}^* \left(\theta + \frac{2\pi j}{n}\right) \tag{23}$$

It is implicitly assumed in the sequel that $d \geq 2$ in $\mathbf{H}_d$, and $m = o(n)$.

Lemma 11 *Under* $\mathbf{H}_0$ *and* $\mathbf{H}_a^{'}$, $\widehat{f^*(\theta)}$ *is consistent, whereas under* $\mathbf{H}_a$, *it diverges to* $+\infty$.

Proof: see the appendix.■

We can now state the main result of this part which gives the asymptotic properties of the statistics under thz alternatives hypotheses.

Theorem 12 *i) Under* $\mathbf{H}_a$: *if* $d \geq 2$ *and* $e(n) = o\left(n^{1/2}\right)$ *for* ξ_n^p, $e(n) = o\left(n^{1/4}\right)$ *for* ξ_n^s, *then* $\left(\frac{n}{e(n)}\right)^2 \times \xi_n^s$ *and* $\left(\frac{n}{e(n)}\right)^2 \times \xi_n^p$ *both converge to a non degenerate law. In particular, if* $\theta \in]0, \pi[$:

$$\left(\frac{n}{e(n)}\right)^2 \times \xi_n^p \Rightarrow 4\sin^2(\theta) \frac{\left|\mathbb{W}_c(1)\right|^2}{\int_0^1 \left|\mathbb{W}_c(t)\right|^2 dt}$$

ii) Under $\mathbf{H}_a^{'}$: *if* $d \geq 4$, $\left|e(n)\right|^2 = o\left(\frac{n}{m}\right)$ *and* $e(n) = o\left(n^{1/4}\right)$ *for* ξ_n^p, $e(n) = o\left(n^{1/8}\right)$ *for* ξ_n^s, *then* ξ_n^p *and* ξ_n^s *diverge to* $+\infty$.

Proof: see the appendix .■

Suppose for example that $e(n) = n^{\alpha}$ and $m = n^{\beta}$ $(\alpha, \beta < 1)$. The conditions in *i)* and *ii)* are fulfilled for ξ_n^p if $\alpha < \min\left(\frac{1}{4}, \frac{1-\beta}{2}\right)$. If we are concerned only with the test of $\mathbf{H}_0$ against $\mathbf{H}_a$, we may take $m = n^{2/3}$ in order to minimize the mean square error of the spectral estimator (Priestley (1988)).

An asymptotic and consistent test with nominal size α for the hypothesis $\mathbf{H}_0$ against the multiple alternative $\mathbf{H}_a \cup \mathbf{H}_a^{'}$ is provided by the following critical region, where $i \in \{p, s\}$:
$W_n^i(\alpha_1, \alpha_2) = \left\{\xi_n^i < c_{\alpha_1}\right\} \cup \left\{\xi_n^i > c_{1-\alpha_2}\right\}$ with $\alpha_1 + \alpha_2 = \alpha$, and c_{α_1} (resp. $c_{1-\alpha_2}$) is the quantile of order α_1(resp. $1 - \alpha_2$) for the limit law of ξ_n^i.

We conclude this section by some qualitative considerations about the choice of the sequence $e(n)$. From the previous results, we have, under $\mathbf{H}_0$:

$$e(n)\,\mathbb{J}_X(\theta_n) - i\sigma C_X\left(\theta, e^{-i\theta}\right)\mathbb{J}_\varepsilon(\theta_n) = O_p\left(\frac{1}{e(n)} + \frac{e(n)}{\sqrt{n}}\right) \tag{24}$$

The convergence of $e(n)\,\mathbb{J}_X(\theta_n)$ to $i\sigma C_X\left(\theta, e^{-i\theta}\right)\mathbb{W}_c(t)$ (and therefore the convergence of ξ_n^p to its limit) depends on $e(n)$ in two opposite contributions:

- The term $1/e(n)$ measures the quality of the (non stochastic) approximation of $f(\theta)$ by $f(\theta_n)$.
- The term $e(n)/\sqrt{n}$ measures the stochastic error made by approximating $e(n)\,\mathbb{J}_X(\theta_n)$ by $e(n)\,C_X\left(e^{-i\theta_n}\right)\mathbb{J}_\varepsilon(\theta_n)$
Observe that $\min\limits_{e(n)}\left(\frac{1}{e(n)}+\frac{e(n)}{\sqrt{n}}\right)=n^{-1/4}$ and the minimum is attained when $e(n)=n^{1/4}$. Now, in order to improve the finite sample properties of the test, we have to decrease the magnitude of the residual term in (24): this is the subject of the following part.

4.3 The case $e(n)=o(n)$

The main idea is to develop the residual term $R_n(\omega,t)$ under $\mathbf{H}_0$ and then modify ξ_n^p in order to allow the improvement $e(n)=o(\sqrt{n})$, and under stronger regularity assumptions, $e(n)=O(n)$.
Remember that $R_n(\omega,t)=-\frac{1}{\sqrt{n}}\left[e^{-i\omega[nt]}Y_n(t,\omega)-Y_n(0,\omega)\right]$ with $Y_n(t,\omega)=C_X(\omega,B)\varepsilon_{[nt]}$

Lemma 13 *i) If f_X is C^4 on $\mathbb{R}$, or if we suppose that $[\mathbf{H}_d, d>2]$, then:*

$$Y_n(t,\theta_n)-Y_n(t,\theta)=o_p(1)\ \textit{ uniformly in } t$$

ii) If f_X is C^5 on $\mathbb{R}$, then $Y_n(t,\theta_n)-Y_n(t,\theta)=O_p\left(\frac{1}{e(n)}\right)$ uniformly in t.

Proof: see the appendix .■
Suppose now that $\theta\neq\pi$. We define:

$$a_n(t)=Y_n(t,\theta_n)-Y_n(t,\theta)$$

Both quantities are $o_p(1)$. Let $u_n=D(\theta,B)\varepsilon_n$ be the real process introduced in lemma 5, which satisfies:

$$X_t=\left(1-2\cos\theta B+B^2\right)u_t$$

We have:

$$\sqrt{n}e^{i\theta_n[nt]}R_n(\theta_n,t)=\underbrace{-a_n(t)+e^{i\theta_n[nt]}a_n(0)}_{b_n(t)}+\left(1-e^{-i\theta}B\right)\left[u_{[nt]}-u_0e^{i\theta_n[nt]}\right]$$

and for $\theta_n\equiv\theta$:

$$\sqrt{n}e^{i\theta[nt]}R_n(\theta,t)=\left(1-e^{-i\theta}B\right)\left[u_{[nt]}-u_0e^{i\theta[nt]}\right]$$

But under $\mathbf{H}_0$, $R_n(\theta,t)=\mathbb{J}_X(\theta,t)$, thus:

$$\sqrt{n}R_n(\theta_n,t)-\sqrt{n}e^{i(\theta-\theta_n)[nt]}\mathbb{J}_X(\theta,t)=e^{-i\theta[nt]}b_n(t)+\left(1-e^{-i\theta}B\right)u_0\left[e^{i(\theta-\theta_n)[nt]}-1\right]$$

We know (see lemma 9) that it is possible to build from $S_j(X,\theta)$ $1 \leq j \leq n$, estimators of u_0 and u_{-1}, $\widehat{u_{0,n}}$ and $\widehat{u_{-1,n}}$ such as:

$$\widehat{u_{0,n}} = u_0 + O_p\left(\frac{1}{\sqrt{n}}\right), \widehat{u_{-1,n}} = u_{-1} + O_p\left(\frac{1}{\sqrt{n}}\right)$$

Define now:

$$\mathbb{J}^*_X(\theta_n) = \mathbb{J}_X(\theta_n) - e^{in(\theta-\theta_n)}\mathbb{J}_X(\theta) - \frac{1}{\sqrt{n}}\left[e^{in(\theta-\theta_n)} - 1\right] \times \left[\widehat{u_{0,n}} - e^{-i\theta}\widehat{u_{-1,n}}\right]$$

Similarly, for $\theta = \pi$, we have with $u_n = C_X(\theta, B)\varepsilon_n$ such as $X_t = (1+B)\,u_t$:

$$\mathbb{J}^*_X(\theta_n) = \mathbb{J}_X(\theta_n) - e^{in(\theta-\theta_n)}\mathbb{J}_X(\theta) - \frac{1}{\sqrt{n}}\left[e^{in(\theta-\theta_n)} - 1\right] \times \widehat{u_{0,n}}$$

Theorem 14 *i) If* $e(n) = O(\sqrt{n})$ *and* f_X *is* C^4, $\mathbb{J}^*_X(\theta_n) - C_X\left(e^{-i\theta_n}\right)\mathbb{J}_\varepsilon(\theta_n) = o_p\left(\frac{1}{\sqrt{n}}\right)$
ii) If $e(n) = n^H$, $H \in [1/2\,, 1[$ *and* f_X *is* C^5 :

$$\mathbb{J}^*_X(\theta_n) - C_X\left(e^{-i\theta_n}\right)\mathbb{J}_\varepsilon(\theta_n) = O_p\left(\frac{1}{n}\right)$$

Proof: *i)* follows immediately from the preceding discussion. For *ii)*, lemma 13 yields
$\mathbb{J}^*_X(\theta_n) - C_X\left(e^{-i\theta_n}\right)\mathbb{J}_\varepsilon(\theta_n) = \frac{1}{\sqrt{n}}O_p\left(\frac{1}{n^H}\right) + O_p\left(\frac{1}{n}\right) = O_p\left(\frac{1}{n}\right)$ since $H \geq 1/2$
■
The theorem yields clearly the convergence:

$$e(n)\mathbb{J}^*_X(\theta_n) \Rightarrow \frac{i\sigma C\left(\theta, e^{-i\theta}\right)}{\sqrt{2}}\mathbb{W}_c(1) \tag{25}$$

and, more interestingly, when assumption *ii)* is fulfilled:

$$e(n)\,\mathbb{J}^*_X(\theta_n) - i\sigma C_X\left(\theta, e^{-i\theta}\right)\mathbb{J}_\varepsilon(\theta_n) = O_p\left(\frac{1}{n^H} + \frac{1}{n^{1-H}}\right) = O_p\left(\frac{1}{n^{1-H}}\right) \tag{26}$$

If we compare this result to (24), we see that the order of magnitude of the residual term is now $O_p\left(\frac{1}{\sqrt{n}}\right)$ for $H = 1/2$, instead of $O_p\left(\frac{1}{n^{1/4}}\right)$ when one uses $\mathbb{J}_X(\theta_n)$ with the "optimal" choice $e(n) = n^{1/4}$. Of course, this result is still valid for $\theta = \pi$.

We now examine $\xi^{p*}_n = \frac{e^2(n)\left|\mathbb{J}^*_X(\theta_n)\right|^2}{\widehat{f_u(\theta)}}$ under $\mathbf{H}_a$.

Theorem 15 *i)If* f_X *is* C^4 *on* $\mathbb{R}$*, then under* $\mathbf{H}_a$, $e(n) = \sqrt{n}$, $m = o(n)$:

$$\xi^{p*}_n = O_p\left(\frac{1}{n}\right)$$

ii) With the hypothesis f_X C^5 *on* $\mathbb{R}$, $e(n) = n^H$, $H \in [1/2\,, 1[$, $m = o(n)$, *we get:* $\xi^{p*}_n = O_p\left(\frac{1}{n^{2-2H}}\right)$.

Under hypothesis $ii)$ with $H = 1/2$, the statistic diverges at rate $1/n$, which is, as expected, very slow compared to the statistic ξ_n^p with $e(n) = n^{1/4}$. Indeed, theorem 12, part ii) shows that $\xi_n^p = O_p\left(\frac{e(n)}{n}\right)^2 = O_p\left(\frac{1}{n\sqrt{n}}\right)$.

Proof: We give details for the case $\theta \neq \pi$ only. First, $\mathbb{J}_X^*(\theta_n) = O_p(1)$. Indeed, $\mathbb{J}_X(\theta_n)$ and $\mathbb{J}_X(\theta)$ are $O_p(1)$, thus:

$$\mathbb{J}_X^*(\theta_n) = O_p(1) - \frac{1}{\sqrt{n}}\left[e^{in(\theta-\theta_n)} - 1\right] \times \left[\widehat{u_{0,n}} - e^{-i\theta}\widehat{u_{-1,n}}\right]$$

Next, $\widehat{u_{0,n}}$ and $\widehat{u_{-1,n}}$ diverge, since, if Y_t is a non-stationary process such as: $(1 - 2\cos\theta B + B^2)Y_t = X_t$, we have with the notations used in lemma 9:

$$a_n = \sqrt{n}\left(1 - \frac{|\Delta_n(2\theta)|^2}{n^2}\right)^{-1} \times \left(\frac{J_Y(\theta)}{n} - \frac{\Delta_n(-2\theta)}{n}\frac{\overline{\mathbb{J}_Y(\theta)}}{n}\right)$$

and from lemma 10:

$$\frac{a_n}{\sqrt{n}} \Rightarrow \mathbb{L}_\theta \equiv -\frac{i\sigma e^{i\theta}}{2\sqrt{2}\sin\theta}C\left(e^{-i\theta}\right)\int_0^1 \mathbb{W}_c(t)dt \tag{27}$$

Next: $\widehat{u_{0,n}} = -2\operatorname{Re} a_n$ and $\widehat{u_{-1,n}} = -2\cos\theta\operatorname{Re} a_n - 2\sin\theta\operatorname{Im} a_n$, so:

$$\frac{\widehat{u_{0,n}} - e^{-i\theta}\widehat{u_{-1,n}}}{\sqrt{n}} \Rightarrow -2i\sin\theta e^{-i\theta}\mathbb{L}_\theta$$

Therefore $\mathbb{J}_X^*(\theta_n) = O_p(1)$. The conclusion follows from theorem 12, part $ii)$ which is insensitive to the choice of the sequence $e(n)$.

■

4.4 The case $e(n) = n$

We adopt in this part a somewhat different approach. Indeed, we examine the behavior of the periodogram when $\theta_n = \theta + t/e(n)$ is considered as a function of t, which will be supposed to lie in $[-2\pi, 2\pi]$ (it could be possible to suppose $t \in [u, v]$ without modifying the results). We assume now that θ_n converges more quickly to θ than in the previous sections, that is $n = O[e(n)]$.

Theorem 16 *Define the following element of* $C[-2\pi, 2\pi]$: $G_n(\theta, t) = \sqrt{2}\mathbb{J}_\varepsilon\left(\theta + \frac{t}{e(n)}\right)$.
Under H_ε, H_ε':
i) $G_n(\theta, .) \Rightarrow \sigma\mathbb{L}_c(\theta, .)$ *in law in* $C[-2\pi, 2\pi]$, *the law of* $\mathbb{L}_c(\theta, .)$ *doesn't depend on* θ.
ii) If $n = o[e(n)]$ *and if* $\theta \neq 0$ *and* π *then* $\mathbb{L}_c(\theta, .) = \mathbb{W}_c(1)$
iii) If $e(n) = n$ *and* $\theta \in]0, \pi[$, $\mathbb{L}_c(\theta, t)$ *is a complex gaussian stationary process with covariance function* $r(h) = 2\frac{\sin(h)}{h} - 4i\frac{\sin^2(h/2)}{h}$. *When* $\theta = 0$ *or* π, $\mathbb{L}_c$ *is stationary, whereas* $\operatorname{Re}\mathbb{L}_c$ *and* $\operatorname{Im}\mathbb{L}_c$ *are gaussian and not stationary.*
iv) If $\widetilde{\theta}$ *is another frequency such as* $\widetilde{\theta} \neq \pm\theta$ *mod.* π:
$\left[G_n(\theta, .), G_n\left(\widetilde{\theta}, .\right)\right]' \Rightarrow \left[\mathbb{L}_c(\theta, .), \mathbb{L}_c\left(\widetilde{\theta}, .\right)\right]'$, *and* $\mathbb{L}_c\left(\widetilde{\theta}, .\right)$ *is independent of* $\mathbb{L}_c(\theta, .)$

Proof: see the appendix .■

From the continuous mapping theorem, if $e(n)$ converges very quickly to infinity, $\sup_{t\in[-2\pi,2\pi]} \left|\mathbb{J}_\varepsilon\left(\theta+\frac{t}{e(n)}\right)-\mathbb{J}_{\varepsilon,n}(\theta)\right| \Rightarrow 0$: this situation is not interesting for our purpose, because there is clearly nothing to be gained by considering such frequencies θ_n. The non-trivial case is $\underline{e(n)=n}$, and from now on, we maintain this assumption. We also assume that $\underline{f \text{ is } C^5}$.

Now suppose that $\theta \notin \{0,\pi\}$ and $f_X(\theta)=0$. Once more, we exploit the residual term:

$$R_n(\omega) = -\frac{1}{\sqrt{n}}\left[e^{-in\omega}Y_n(\omega)-Y_0(\omega)\right] \text{ with} Y_n(\omega) = C_X(\omega,B)\varepsilon_n$$

From lemma (13), $Y_n(\theta_{n,t})-Y_n(\theta)=O_p\left(\frac{1}{n}\right)$. Now, if $\theta\neq\pi$ we define:
$a_n(t)=Y_n(\theta_{n,t})-Y_n(\theta)$, $\widetilde{a}_n(t)=Y_0(\theta_{n,t})-Y_0(\theta)$ and the real process $u_n=D(\theta,B)\varepsilon_n$ such as:

$$X_t = \left(1-2\cos\theta B+B^2\right)u_t$$

$$\sqrt{n}e^{in\theta_{n,t}}R_n(\theta_{n,t}) = \underbrace{-a_n(t)+e^{in\theta_{n,t}}a_n(0)}_{b_n(t)} + \left(1-e^{-i\theta}B\right)\left[u_n-u_0e^{in\theta_{n,t}}\right]$$

$\sup|b_n(t)|=O_p\left(\frac{1}{n}\right)$ and for $\theta_n\equiv\theta$, $\sqrt{n}e^{in\theta}R_n(\theta)=\left(1-e^{-i\theta}B\right)\left[u_n-u_0e^{in\theta}\right]$.
But under $\mathbf{H_0}$, $R_n(\theta)=\mathbb{J}_X(\theta)$, thus:

$$\sqrt{n}R_n(\theta_{n,t})-\sqrt{n}e^{in(\theta-\theta_n)}\mathbb{J}_X(\theta) = e^{-in\theta}b_n(t)+\left(1-e^{-i\theta}B\right)u_0\left[e^{in(\theta-\theta_n)}-1\right]$$

With $\theta_{n,t}=\theta+\frac{t}{n}$:

$$\sqrt{n}R_n(\theta_{n,t})-\sqrt{n}e^{-it}\mathbb{J}_X(\theta) = e^{-in\theta}b_n(t)+\left[e^{-it}-1\right]\times\left[u_0-e^{-i\theta}u_{-1}\right]$$

and $b_n(t)=-a_n(t)+e^{in\theta}e^{it}a_n(0)$
Estimators of u_0 and u_{-1}, $\widehat{u_{0,n}}$ and $\widehat{u_{-1,n}}$ previously discussed satisfy:

$$\widehat{u_{0,n}} = u_0+O_p\left(\frac{1}{\sqrt{n}}\right), \widehat{u_{-1,n}} = u_{-1}+O_p\left(\frac{1}{\sqrt{n}}\right)$$

Let the "corrected" Fourier transform be:

$$\mathbb{J}_X^*(\theta_{n,t}) = \mathbb{J}_X(\theta_{n,t})-e^{-it}\mathbb{J}_X(\theta)-\frac{1}{\sqrt{n}}\left[e^{-it}-1\right]\times\left[\widehat{u_{0,n}}-e^{-i\theta}\widehat{u_{-1,n}}\right] \tag{28}$$

$\mathbb{J}_X^*(\theta_{n,0})=0$ and:

$$\begin{aligned} n\mathbb{J}_X^*(\theta_{n,t}) &= nC_X\left(e^{-i\theta_{n,t}}\right)\mathbb{J}_\varepsilon(\theta_{n,t})+\sqrt{n}e^{-in\theta}b_n(t) \\ &+\left[e^{-it}-1\right]\times\underbrace{\sqrt{n}\left[\widehat{u_{0,n}}-u_0-e^{-i\theta}\left(\widehat{u_{-1,n}}-u_{-1}\right)\right]}_{C_n} \end{aligned}$$

We know that:

$$\begin{aligned} \sqrt{n}(a_n-a) &= \left(1-\frac{|\Delta_n(\theta)|^2}{n^2}\right)^{-1}\times\left(\mathbb{J}_u(\theta)-\frac{\Delta_n(-\theta)}{n}\overline{\mathbb{J}_u(\theta)}\right) \\ &= \mathbb{J}_u(\theta)+O_p\left(\frac{1}{n}\right) \end{aligned}$$

and $\widehat{u_{0,n}} = -2\operatorname{Re} a_n$ and $\widehat{u_{-1,n}} = -2\cos\theta \operatorname{Re} a_n - 2\sin\theta \operatorname{Im} a_n$, Thus:

$$\sqrt{n}\left[\widehat{u_{0,n}} - u_0 - e^{-i\theta}\left(\widehat{u_{-1,n}} - u_{-1}\right)\right] = -2i\sin\theta e^{-i\theta}\mathbb{J}_u(\theta) + O_p\left(\frac{1}{n}\right)$$

with $\mathbb{J}_u(\theta) = D\left(\theta, e^{-i\theta}\right)\mathbb{J}_\varepsilon(\theta_n) + O_p\left(\frac{1}{\sqrt{n}}\right)$ and $D\left(\theta, e^{-i\theta}\right) = -\left(1 - e^{-2i\theta}\right)^{-1} C_X\left(\theta, e^{-i\theta}\right)$

Lastly, $nC_X\left(e^{-i\theta_{n,t}}\right) = n\left(e^{-it/n} - 1\right) C_X\left(\theta, e^{-i\theta_{n,t}}\right)$ converge to $-itC_X\left(\theta, e^{-i\theta}\right)$ uniformly in t.

From the continuous mapping theorem and theorem (16) we get:

$$n\mathbb{J}_X^*(\theta_{n,t}) \Rightarrow \frac{\sigma}{\sqrt{2}} C_X\left(\theta, e^{-i\theta}\right)\mathbb{V}(t) \tag{29}$$

with $\mathbb{V}$ gaussian complex process defined by:

$$\mathbb{V}(t) = -it\mathbb{L}_c(\theta, t) + \left[1 - e^{-it}\right]\mathbb{L}_c(\theta, 0) \tag{30}$$

If $r(h)$ is the covariance function of $\mathbb{L}_c(t)$, we note that:
$\mathbb{E}\left|\mathbb{V}(t)\right|^2 = 2\operatorname{Re}\left\{it\left[1 - e^{-it}\right] r(t)\right\} + \left\{\left|1 - e^{-it}\right|^2 + t^2\right\} r(0)$

and then:

$$\mathbb{E}\left|\mathbb{V}(t)\right|^2 = 4(1 - \cos t)(1 - 2\cos t) + 2t^2$$

We remark that:

$$n\mathbb{J}_X^*(\theta_{n,t}) - i\sigma C_X\left(\theta, e^{-i\theta}\right)\mathbb{J}_\varepsilon(\theta_{n,t}) - \left[e^{-it} - 1\right] C_n = O_p\left(\frac{1}{n}\right) + O_p\left(\frac{1}{\sqrt{n}}\right) = O_p\left(\frac{1}{\sqrt{n}}\right)$$

The order of the "residual" term is then $1/\sqrt{n}$.

Now suppose that $f_X(\theta) > 0$ (and $\theta \notin \{0, \pi\}$).

$$\begin{aligned}
\mathbb{J}_X^*(\theta_{n,t}) &= C_X\left(e^{-i\theta_{n,t}}\right)\mathbb{J}_\varepsilon(\theta_{n,t}) - 1/n \times e^{-it} C_X\left(e^{-i\theta}\right)\mathbb{J}_\varepsilon(\theta) \\
&\quad + 1/\sqrt{n} \times e^{-in\theta} b_n(t) + \left[e^{-it} - 1\right] \times 1/\sqrt{n} \times \left[\widehat{u_{0,n}} - u_0 - e^{-i\theta}\left(\widehat{u_{-1,n}} - u_{-1}\right)\right]
\end{aligned}$$

$$\begin{aligned}
\sup_{0\le t\le 1}\left|\mathbb{J}_X^*(\theta_{n,t})\right| &\le \sup_{0\le t\le 1}\left|C_X\left(e^{-i\theta_{n,t}}\right)\mathbb{J}_\varepsilon(\theta_{n,t})\right| + C\left|\mathbb{J}_\varepsilon(\theta)\right| \\
&\quad + O_p\left(\frac{1}{n\sqrt{n}}\right) + \frac{2}{\sqrt{n}}\left|\widehat{u_{0,n}} - u_0 - e^{-i\theta}\left(\widehat{u_{-1,n}} - u_{-1}\right)\right|
\end{aligned}$$

(27) and theorem (16) imply that $\sup_{0\le t\le 1}\mathbb{J}_X^*(\theta_{n,t}) = O_p(1)$.

From the preceding developments, it is natural to introduce the following stochastic function:

$$\widetilde{\xi}_n^p(t) = \frac{n^2\left|\mathbb{J}_X^*\left(\theta + \frac{t}{n}\right)\right|^2}{\widehat{f_u(\theta)}}$$

We summarize our results in the following theorem:

Theorem 17 *If f_X is C^5 on $\mathbb{R}$, $m = o(n)$, then under $\mathbf{H}_0$, $\widetilde{\xi}_n^p(t) \Rightarrow 4\pi\sin^2\theta\left|\mathbb{V}(t)\right|^2$ and under $\mathbf{H}_a$, $\widetilde{\xi}_n^p(t) = O_p(1)$.*

This result, although interesting from a theoretical point of view, does not directly provide a test of $\mathbf{H}_0$ since $\widetilde{\xi}_n^p(t)$ does not diverge under $\mathbf{H}_a$. But it is in fact straightforward to define an asymptotic consistent test, which moreover avoids the calculation of $\widehat{u_0}$ and $\widehat{u_{-1}}$. We proceed as follows. Under $\mathbf{H}_0$, from (29) and (30), we obtain the limiting distribution of the slope of $\mathbb{J}_X^*$ between $\theta - \frac{\pi}{n}$ and $\theta + \frac{\pi}{n}$:

$$n\left[\mathbb{J}_X^*\left(\theta + \frac{\pi}{n}\right) - \mathbb{J}_X^*\left(\theta - \frac{\pi}{n}\right)\right] \Rightarrow -\frac{i\sigma\pi}{\sqrt{2}} C_X\left(\theta, e^{-i\theta}\right)\left[\mathbb{L}_c(\theta,\pi) - \mathbb{L}_c(\theta,-\pi)\right] \quad (31)$$

The proof of theorem 16 shows that for all $\theta \in \left]0,\pi\right]$:

$$\mathbb{L}_c(\theta,\pi) - \mathbb{L}_c(\theta,-\pi) \rightsquigarrow \mathbb{N}_c(0,4)$$

It yields $\left|\mathbb{L}_c(\theta,2\pi) - \mathbb{L}_c(\theta,-2\pi)\right|^2 \equiv 2\chi_2(2)$. Now, it is easily seen that:

$$\mathbb{J}_X^*\left(\theta + \frac{\pi}{n}\right) - \mathbb{J}_X^*\left(\theta - \frac{\pi}{n}\right) = \mathbb{J}_X\left(\theta + \frac{\pi}{n}\right) - \mathbb{J}_X\left(\theta - \frac{\pi}{n}\right)$$

Under $\mathbf{H}_a$ we obtain the convergence:

$$\begin{aligned}
\mathbb{J}_X\left(\theta + \tfrac{\pi}{n}\right) - \mathbb{J}_X\left(\theta - \tfrac{\pi}{n}\right) &= C_X\left(e^{-i\theta_{n,\pi}}\right)\mathbb{J}_\varepsilon(\theta_{n,\pi}) - C_X\left(e^{-i\theta_{n,-\pi}}\right)\mathbb{J}_\varepsilon(\theta_{n,-\pi}) + O_p\left(\tfrac{1}{\sqrt{n}}\right) \\
&= \underbrace{\left[C_X\left(e^{-i\theta_{n,\pi}}\right) - C_X\left(e^{-i\theta_{-n,\pi}}\right)\right]}_{O(n^{-1})}\mathbb{J}_\varepsilon(\theta_{n,-\pi}) \\
&\quad + C_X\left(e^{-i\theta_{n,\pi}}\right)\left[\mathbb{J}_\varepsilon(\theta_{n,\pi}) - \mathbb{J}_\varepsilon(\theta_{n,-\pi})\right] + O_p\left(\tfrac{1}{\sqrt{n}}\right) \\
&\Rightarrow C_X\left(e^{-i\theta}\right)\left[\mathbb{L}_c(\theta,\pi) - \mathbb{L}_c(\theta,-\pi)\right]
\end{aligned}$$

We define the following statistic:

$$\xi_n^{s*} = \frac{n^2\left|\mathbb{J}_X\left(\theta + \frac{\pi}{n}\right) - \mathbb{J}_X\left(\theta - \frac{\pi}{n}\right)\right|^2}{\left[\widehat{f_X(\theta)} + \widehat{f_u(\theta)}^{-1}\right]^{-1}}$$

The numerator of ξ_n^{s*} has been studied among the preceding lines. We now have to study the behavior of the denominator. It is clear that:

Under $\mathbf{H}_0$: $\widehat{f_X(\theta)} \xrightarrow{P} 0$ and $\widehat{f_u(\theta)} \xrightarrow{P} f_u(\theta) > 0$ thus $\left[\widehat{f_X(\theta)} + \widehat{f_u(\theta)}^{-1}\right]^{-1} \xrightarrow{P} f_u(\theta)$

Under $\mathbf{H}_a$: $\widehat{f_X(\theta)} \xrightarrow{P} f_X(\theta) > 0$ and $\widehat{f_u(\theta)}^{-1} \xrightarrow{P} 0$ thus:

$$\left[\widehat{f_X(\theta)} + \widehat{f_u(\theta)}^{-1}\right]^{-1} \xrightarrow{P} f_X(\theta)^{-1}$$

Hence, we get (the case $\theta \in \{0,\pi\}$ identical):

Theorem 18 *If f_X is C^5 on $\mathbb{R}$, $m = o(n)$, then:*
Under $\mathbf{H}_0$:

$$\begin{aligned}
\xi_n^{s*} &\Rightarrow 8\pi^3 \sin^2\theta \chi_2(2) \ \ \textit{if } \theta \in \left]0,\pi\right[\\
\xi_n^{s*} &\Rightarrow 2\pi^3 \chi_2(2) \ \ \textit{if } \theta \in \{0,\pi\}
\end{aligned} \quad (32)$$

Under $\mathbf{H}_a$:

$$\frac{\xi_n^{s*}}{n^2} \Rightarrow 4\pi f_X^2(\theta)\chi_2(2) \quad (33)$$

By replacing the term $\frac{\pi}{n}$ by $\frac{2\pi}{n}$, we may also consider the following statistic:

$$\xi_n^{s**}=\frac{n^2\left|\mathbb{J}_X^*\left(\theta+\frac{2\pi}{n}\right)-\mathbb{J}_X^*\left(\theta-\frac{2\pi}{n}\right)\right|^2}{\left[\widehat{f_X(\theta)}+\widehat{f_u(\theta)}^{-1}\right]^{-1}}=\frac{n^2\left|\mathbb{J}_X\left(\theta+\frac{2\pi}{n}\right)-\mathbb{J}_X\left(\theta-\frac{2\pi}{n}\right)\right|^2}{\left[\widehat{f_X(\theta)}+\widehat{f_u(\theta)}^{-1}\right]^{-1}}$$

It is straightforward to check that, for $\theta\in\,]0,\pi[\,$, under $\mathbf{H}_0$, $\xi_n^{s**}\Rightarrow 16\pi^3\sin^2\theta\chi_2\left(2\right)$, and under $\mathbf{H}_a$, $\frac{\xi_n^{s**}}{n^2}\Rightarrow 8\pi f_X^2(\theta)\chi_2\left(2\right)$. This statistic is designed to accommodate for non-zero intercepts in the process X_t, as it will be seen later.

5 A test free of nuisance parameter

It is indeed very easy to derive such a test, through a slight extension of theorem 7.

Theorem 19 *If $\theta_n,\widetilde{\theta}_n\in\,]0,\pi[$ are two sequences converging to θ, $\theta_n=\theta+e^{-1}(n)$, $\widetilde{\theta}_n=\theta+\widetilde{e}^{-1}(n)$ and such as $\forall n$, $\theta_n\neq\widetilde{\theta}_n$, and:*
either $\widetilde{e}(n)=o\left(n\right)$ and $\widetilde{e}(n)=o\left[e\left(n\right)\right]$.
either $\widetilde{e}(n)=e\left(n\right)\left[1+\lambda_n\right]$ with $\lambda_n=o\left(1\right)$ and $\widetilde{e}(n)=o\left(n\lambda_n\right)$.
We define $S_n(t,\theta_n)=\sqrt{\frac{2}{n}}\left(\sum_{k=1}^{[nt]}\exp(ik\theta_n)\varepsilon_k,\sum_{k=1}^{[nt]}\exp(ik\widetilde{\theta}_n)\varepsilon_k\right)'$ and $T_n(t,\theta_n)$ the appropriate interpolation of $S_n(t,\theta_n)$ which belongs to $C\left[0,1\right]$.
Then $T_n(t,\theta_n)\underset{n\to\infty}{\Rightarrow}\sigma\left(\mathbb{W}_{c,1}(t),\mathbb{W}_{c,2}(t)\right)'$. $\mathbb{W}_{c,1}(t)$ are $\mathbb{W}_{c,2}(t)$ two independent complex Brownian motions.

Proof: see the appendix .■

Suppose now that $\mathbf{H}$ is fulfilled with $d\geq 2$. Under $\mathbf{H_0}$ and conditions

$$e(n)=o\left(n^{1/2}\right),\widetilde{e}(n)=o\left(n^{1/2}\right)$$

we get easily, as in theorem 8:

$$\begin{pmatrix} e^2(n)\mathbb{I}_X\left(\theta_n\right)\\ \widetilde{e}^2(n)\mathbb{I}_X\left(\widetilde{\theta}_n\right)\end{pmatrix}\Rightarrow\begin{pmatrix}\sigma^2\left|C_X\left(\theta,e^{-i\theta}\right)\right|^2\chi\left(2\right)\\ \sigma^2\left|C_X\left(\theta,e^{-i\theta}\right)\right|^2\widetilde{\chi}\left(2\right)\end{pmatrix}\tag{34}$$

The chi-square are indépendent from the preceding theorem.

We then deduce $\left(\frac{e(n)}{\widetilde{e}(n)}\right)^2\frac{\mathbb{I}_X(\theta_n)}{\mathbb{I}_X\left(\widetilde{\theta}_n\right)}\Rightarrow\frac{\chi(2)}{\widetilde{\chi}(2)}\equiv F_{2,2}$, Fisher distribution with (2,2) degrees of freedom.

This law has no expectation. For that reason, we prefer state the result in the following form:

$$\xi_n^r=\left|\frac{e(n)}{\widetilde{e}(n)}\right|\sqrt{\frac{\mathbb{I}_X\left(\theta_n\right)}{\mathbb{I}_X\left(\widetilde{\theta}_n\right)}}\Rightarrow\sqrt{F_{2,2}}\tag{35}$$

The c.d.f of $\sqrt{F_{2,2}}$ is, for $x\geq 0$, $F(x)=\frac{x^2}{1+x^2}$. An asymptotic test with size α of $\mathbf{H}_0\left(s\right)$ against $\mathbf{H}_a$ is:

$W_n(\alpha) = \{\xi_n^r(s) > c_{1-\alpha}\}$ with $c_{1-\alpha} = \sqrt{\frac{1-\alpha}{\alpha}}$ quantile of order $1-\alpha$ for the law $\sqrt{F_{2,2}}$
The test is consistent since $\widetilde{e(n)} = o(e(n))$ and under $\mathbf{H}_a$:

$$\sqrt{\frac{\mathbb{I}_X(\theta_n)}{\mathbb{I}_X\left(\widetilde{\theta}_n\right)}} \Rightarrow \sqrt{F_{2,2}}$$

thus $\xi_n^r(s) \xrightarrow{P} +\infty$.
We note that $\mathbb{E}\sqrt{F_{2,2}} = \frac{\pi}{2}$ and var $\sqrt{F_{2,2}} = \infty$. $\xi_n^r(s)$ does not admit (asymptotically) moments of order > 1. As a consequence, under $\mathbf{H}_0$, outliers are likely to occur more often with ξ_n^r than with $\xi_n^p, \xi_n^s, \xi_n^{p*}$.
One possible choice is $e(n) = \frac{n^\varsigma}{C}$ $(\varsigma < \frac{1}{2})$ and $\widetilde{e(n)} = \frac{\log(n)}{\widetilde{C}}$, C and $\widetilde{C}$ constant

Suppose now $\theta \in]0,\pi[$, and $e(n) = n$, $\widetilde{e}(n) = \sqrt{n}$. From (25), (31) and theorem 19:

$$\begin{pmatrix} \sqrt{n}\mathbb{J}_X^*(\theta_n) \\ n\left[\mathbb{J}_X\left(\theta + \frac{\pi}{n}\right) - \mathbb{J}_X\left(\theta - \frac{\pi}{n}\right)\right] \end{pmatrix} \Rightarrow \frac{i\sigma C\left(\theta, e^{-i\theta}\right)}{\sqrt{2}} \begin{pmatrix} \mathbb{W}_c(1) \\ \sqrt{2}\pi\widetilde{\mathbb{W}}_c(1) \end{pmatrix}$$

with $\mathbb{J}_X^*(\theta_n) = \mathbb{J}_X\left(\theta + \frac{\pi}{\sqrt{n}}\right) - e^{-i\pi\sqrt{n}}\mathbb{J}_X(\theta) - \frac{1}{\sqrt{n}}\left[e^{-i\pi\sqrt{n}} - 1\right] \times \widehat{u_{0,n}}$
and $\sqrt{2}\pi\widetilde{\mathbb{W}}_c(1) \equiv [\mathbb{L}_c(\theta,\pi) - \mathbb{L}_c(\theta,-\pi)]$. From the continuous mapping theorem:

$$\xi_n^{r*} = \sqrt{n}\frac{\left|\mathbb{J}_X\left(\theta + \frac{\pi}{n}\right) - \mathbb{J}_X\left(\theta - \frac{\pi}{n}\right)\right|}{\left|\mathbb{J}_X^*(\theta_n)\right|} \Rightarrow \sqrt{2}\pi\sqrt{F_{2,2}}$$

A consistent test of $\mathbf{H}_0$ against $\mathbf{H}_a$ with size α is then:

$$W_n(\alpha) = \left\{\xi_n^{r*}(s) > \sqrt{\frac{1-\alpha}{\alpha}}\right\}$$

6 Extensions and applications

6.1 Non-zero mean processes

Suppose now that we observe the process $Y_t = \text{Re}\left(ce^{i\nu t}\right) + X_t$ instead of X_t, where X_t is the zero-mean process considered previously and ν some known[4] frequency in $[0,\pi]$. In other words, we include a "seasonal" intercept to the dynamic of X_t. We know (see Brillinger (1981)) that a preliminary regression of Y_t on $\cos(\nu t)$ and $\sin(\nu t)$ provides an estimate of c, $\widehat{c}$ such as:

- Under $\mathbf{H}_0$, $\widehat{c} - c = O_p(1/n)$ if $\nu = \theta$; if $\nu \neq \theta$, $\widehat{c} - c = O_p(1/\sqrt{n})$ and $\widehat{c}$ is asymptotically normal.
- Under $\mathbf{H}_a$, $\widehat{c} - c = O_p(1/\sqrt{n})$ and $\widehat{c}$ is asymptotically normal.

[4] We do not adress here the problem which occurs when θ_1 is not known, and therefore must be estimated.

Note that we can not test the hypothesis $c = 0$ when $\nu = \theta$ with standard techniques. Let $\widehat{X}_t$ be the residual of this regression, and $d = \overline{c}$:

$$\mathbb{J}_X(\omega) - \mathbb{J}_{\widehat{X}}(\omega) = \frac{\widehat{c} - c}{\sqrt{n}} \Delta_n(\nu - \omega) + \frac{\widehat{d} - d}{\sqrt{n}} \Delta_n(\nu + \omega) \tag{36}$$

To save space, we only consider the tests ξ_n^p and ξ_n^{s*}.

- Test ξ_n^p
  If $\nu = \theta$, we get $\left|\mathbb{J}_X(\theta_n) - \mathbb{J}_{\widehat{X}}(\theta_n)\right| \leq \frac{|\widehat{c}-c|}{\sqrt{n}} \times e(n) + \frac{\left|\widehat{d}-d\right|}{\sqrt{n}} \times O(1)$
  So under $\mathbf{H_0}$, $e(n)\left[\mathbb{J}_X(\theta_n) - \mathbb{J}_{\widehat{X}}(\theta_n)\right] = O_p\left(\frac{e^2(n)}{n\sqrt{n}}\right)$
  and under $\mathbf{H}_a$, $\left[\mathbb{J}_X(\theta_n) - \mathbb{J}_{\widehat{X}}(\theta_n)\right] = O_p\left(\frac{e(n)}{n}\right)$
  If $\nu \neq \theta$, we get $\left|\mathbb{J}_X(\theta_n) - \mathbb{J}_{\widehat{X}}(\theta_n)\right| = \frac{|\widehat{c}-c|}{\sqrt{n}} \times O(1) + \frac{\left|\widehat{d}-d\right|}{\sqrt{n}} \times O(1)$
  So under $\mathbf{H_0}$, $e(n)\left[\mathbb{J}_X(\theta_n) - \mathbb{J}_{\widehat{X}}(\theta_n)\right] = O_p\left(\frac{e(n)}{\sqrt{n}}\right)$
  and under $\mathbf{H}_a$, $\left[\mathbb{J}_X(\theta_n) - \mathbb{J}_{\widehat{X}}(\theta_n)\right] = O_p\left(\frac{1}{n}\right)$
  We see that if $e(n) = o(\sqrt{n})$, the limit of the numerator of ξ_n^p is not affected by the replacement of X_t by $\widehat{X}_t$.

- Test ξ_n^{s**}
  Suppose first $\nu \neq \theta$.
  Let $J_n = \mathbb{J}_X\left(\theta + \frac{2\pi}{n}\right) - \mathbb{J}_X\left(\theta - \frac{2\pi}{n}\right)$ and $\widetilde{J_n} = \mathbb{J}_{\widehat{X}}\left(\theta + \frac{2\pi}{n}\right) - \mathbb{J}_{\widehat{X}}\left(\theta - \frac{2\pi}{n}\right)$
  $J_n - \widetilde{J_n} = \frac{\widehat{b}-b}{\sqrt{n}}\left[\Delta_n\left(\theta - \nu + \frac{2\pi}{n}\right) - \Delta_n\left(\theta - \nu - \frac{2\pi}{n}\right)\right]$
  Now, from (3), $J_n - \widetilde{J_n} = O\left(\frac{1}{n}\right)$ under $\mathbf{H_0}$ and $\mathbf{H}_a$. Thus:

$$\begin{aligned} n\left[J_n - \widetilde{J_n}\right] &= O_p(1) \text{ under } \mathbf{H_0} \\ J_n - \widetilde{J_n} &= O_p\left(\tfrac{1}{n}\right) \text{ under } \mathbf{H}_a \end{aligned} \tag{37}$$

Now, if $\nu = \theta$, one obtains simply $J_n = \widetilde{J_n}$ because $\Delta_n\left(\frac{2\pi}{n}\right) = \Delta_n\left(-\frac{2\pi}{n}\right) = 0$

It follows that the limit of the numerator of ξ_n^{s*} is not affected by the replacement of X_t by $\widehat{X}_t$ if $\nu = \theta$. However, when $\nu \neq \theta$, nothing can be said from (37). But fortunately, applying the operator $(1 - 2\cos vB + B^2)$ to Y_t solves the problem[5], and do not affect our test, thanks to its local properties.

We turn now to the denominator of ξ_n^p and ξ_n^{s**}. For a stationary process around deterministic terms, a classical result asserts that filtering deterministic components by running preliminary OLS leaves the limiting properties of spectral estimators unchanged (see e.g Priestley (1988)). But we must verify that this result is still valid when the cumulated series is involved, for the calculation of $\widehat{f_u(\theta)}$ under $\mathbf{H_0}$.

Lemma 20 *Under* $\mathbf{H_0}$*,* $\widehat{f_u(\theta)}$ *is still a consistent estimator of* $f_u(\theta)$ *if* X_t *is replaced by* $\widehat{X_t}$

[5] In this case, we can also test the hypothesis $c = 0$ by standard methods.

Proof: see the appendix.■

As a conclusion, once the process has been properly filtered, the seasonal intercept at frequency θ is, at least asymptotically, without effect on our test statistics. This result could be extended to more complex deterministic terms (linear and quadratic seasonal at the frequency θ), but this will not be done here. By contrast, this result is invalidated when linear terms at a frequency $\theta_1 \neq \theta$ remain in the data. However, this situation can be handled in a simple way for the generic case where this component takes the form:

$$d_t = \sum_{k=0}^{P} \left(\lambda_k t^k \cos(\theta_1 t) + \mu_k t^k \sin(\theta_1 t) \right)$$

where typically $P = 0$ or 1 and θ_1 is known. Because $\left(1 - 2\cos\theta_1 B + B^2\right)^{P+1} d_t = 0$, it suffices to difference the series as much as necessary to get rid of the influence of these terms. Of course, this does not affect the tests for the frequency θ.

6.2 Testing the null of stationarity against seasonal unit-roots alternatives

The model is the following[6]:

$$\begin{aligned} Y_t &= \sum_{\theta_u \in F} \left[(a_u \cos t\theta_u + b_u \sin t\theta_u) + t\left(a_u' \cos t\theta_u + b_u' \sin t\theta_u \right) \right] + X_t \\ &\prod_{\theta_u \in F} \left(1 - 2\cos\theta_u B + B^2\right)^{d_u} X_t = \varepsilon_t \end{aligned} \quad (38)$$

F is a finite part of $[0, \pi]$, and we make the following assumptions:

$\mathbf{H}_1 : d_u \in \{0, 1\}$ for $\theta_u \in]0, \pi[\, , d_u \in \{0, 1/2\}$ for $\theta_u \in \{0, \pi\}$
$\mathbf{H}_2 : \varepsilon_t$ is a stationary ARMA process with stricly positive spectrum at frequencies $\theta_u \in F$.

Model (38) allows for quite different seasonal representations, deterministic or stochastic, slowly changing or highly unstable. Moreover, it has the desirable property that the parameters of deterministic components keep the same meaning whatever the value of d_u is: this property is not shared by the model considered by Engle *et alii* (1990). The hypothesis $\mathbf{H}_1$ means that integrated process of order two at some frequency $\theta_u \in F$ are excluded. Now, the problem at hand is the identification of d_u, or more precisely, to decide between $d_u = 0$ and $d_u = 1$.

Let θ_{u_0} be fixed. We define the operators:

$$Z_{u_0}^1(B) = \prod_{\theta_u \in F - \{\theta_{u_0}\}} \left(1 - 2\cos\theta_u B + B^2\right)^2$$

$$Z_{u_0}^2(B) = \left(1 - 2\cos\theta_{u_0} B + B^2\right)$$

We get immediately:

$$Z_{u_0}^1(B) Y_t = Z_{u_o}^1(B) \left[(a_{u_0} \cos t\theta_{u_0} + b_{u_0} \sin t\theta_{u_0}) + t\left(a_{u_0}' \cos t\theta_{u_0} + b_{u_0}' \sin t\theta_{u_0} \right) \right] + Z_{u_o}^1(B) X_t$$

[6] Implicitely, $b_u = b_u' = 0$ when $\theta_u = 0$ or π.

$$Z_{u_0}^2(B) Z_{u_o}^1(B) Y_t = Z_{u_0}^1(B)\left(a_{u_0}^{'}\left[\cos t\theta_{u_0} - \cos(t-2)\theta_{u_0}\right] + b_{u_0}^{'}\left[\sin t\theta_{u_0} - \sin(t-2)\theta_{u_0}\right]\right)$$
$$+ Z_{u_0}^2(B) Z_{u_o}^1(B) X_t$$

and the filtered variable:

$$\widetilde{X}_t = Z_{u_0}^2(B) Z_{u_0}^1(B) X_t = \left(\prod_{\theta_u \in F-\{\theta_{u0}\}} \left(1 - 2\cos\theta_u B + B^2\right)^{2-d_u}\right) \times \left(1 - 2\cos\theta_{u_o} B + B^2\right)^{1-d_{u0}} \varepsilon_t$$

Testing $d_{u_0} = 0$ against $d_{u_0} = 1$ is equivalent to:

$$f_{\widetilde{X}}(\theta_{u_0}) = 0 \text{ against } f_{\widetilde{X}}(\theta_{u_0}) > 0 \tag{39}$$

Note that the null hypothesis is stationarity at frequency θ_{u_0}. We remark that:

$$W_t = Z_{u_o}^1(B)\left(a_{u_0}^{'}\left[\cos t\theta_{u_0} - \cos(t-2)\theta_{u_0}\right] + b_{u_0}^{'}\left[\sin t\theta_{u_0} - \sin(t-2)\theta_{u_0}\right]\right)$$
$$= T_t^{'}\beta$$

for some known deterministic vector T_t and unknown coefficients $\beta = \left(a_{u_0}^{'}, b_{u_0}^{'}\right)^{'}$. The test procedure is then:

- Run a preliminary regression of $Z_{u_0}^2(B) Z_{u_o}^1(B) Y_t$ on T_t.
- Get the residual of this regression, and then perform the test of nullity of the spectrum at frequency θ_{u_0}.

Moreover, it is clear that the test-statistics associated with two distinct frequencies, θ_u and θ_v are asymptotically independent, because the Brownian motions which occur in their respective limit law are themselves independent.

Remark 5 *Note that these tests are robust to the presence of conditional heteroskedasticity in ε_t, a result which does not seem to have been established for Dickey-Fuller or Phillips-Perron tests (see Phillips and Perron (1988)). Indeed, to the knowledge of the author, the assumption of mixing which is needed to find the limiting distribution of these tests is lacking for GARCH processes.*

6.3 The case θ unknown

Even if our main goal is to test wether the spectrum vanishes at seasonal frequencies, it may be interesting to consider the situation where θ is not known in advance. The problem becomes now:

$$\mathbf{H}_0 : \inf f(\omega) = 0$$
$$\mathbf{H}_a : \inf f(\omega) > 0$$

We define $\theta = \inf\{\omega \in [0,\pi], f(\omega) = \inf f\}$. Unfortunately, the tests developed in this paper are not directly helpful to achieve this task. Consider for example our first test statistic:

$$\xi_n^p(\theta) = \frac{e^2(n)\mathbb{I}_X(\theta_n)}{\widehat{f_u(\theta)}}$$

Though $\xi_n^p(\theta)$ is a continuous function of θ, this is not the case for its limit law $\xi^p(\theta)$ since:

$$\theta_1 \neq \theta_2 \Rightarrow \xi^p(\theta_1) \text{ is independent of } \xi^p(\theta_2)$$

This means that the periodogram fails to provide global information relative to the spectrum. As an illustration of this well-known result, let us consider the statistic:

$$\xi_n^i = \inf_{0 \leq j \leq [n/2]} \mathbb{I}_X\left(\frac{2\pi j}{n}\right)$$

It converges to zero, even when $f_X(\omega)$ is strictly positive everywhere. In the simplest case $X_t = \varepsilon_t$ i.i.d $N(0,1)$ the variables $\frac{1}{2}\mathbb{I}_X\left(\frac{2\pi j}{n}\right)$ are i.i.d following a χ_2 distribution. Then:
$P\left(\xi_n^i > x\right) = e^{-\frac{x}{2}([n/2]+1)}$ and $P\left(n\xi_n^i > x\right) \to e^{-\frac{x}{4}}$, thus $\xi_n^i = O_p\left(\frac{1}{n}\right)$.

The strategy is then to get an estimate $\widehat{\theta}$ of θ and to plug it into our test statistics. Here, we follow Müller and Prewitt ((1992)). We suppose[7] that ε_t is gaussian iid, $\mathbb{H}_d$ is satisfied with $d = 4$, $f^{(2)}(\theta) > 0$ and $\theta \in]0,\pi[$. This last hypothesis is rather unsatisfactory, but is needed to overcome the discontinuity which arises at the points 0 and π for the limit law.
Then from Müller and Prewitt ((1992)), one gets[8]

$$\begin{array}{l} \text{Under } \mathbf{H}_0 : n^{\frac{2}{7}}\left(\widehat{\theta} - \theta\right) \xrightarrow{P} 0 \\ \text{Under } \mathbf{H}_a : n^{\frac{2}{7}}\left(\widehat{\theta} - \theta\right) \xrightarrow{P} N\left(c, \sigma^2\right) \end{array}$$

We do not need to be more specific about $\widehat{\theta}$, and the parameter (c, σ^2). We just recall that $\widehat{\theta}$ is the arg-min of some well-defined kernel estimator of f. We introduce a slight modification of this estimator (remember that $\widehat{\theta} \in [0,\pi]$):

$$\widehat{\theta}^* = \max\left(\widehat{\theta}, \frac{1}{n^\delta}\right) \text{ if } \widehat{\theta} < \frac{\pi}{2}, \widehat{\theta}^* = \min\left(\widehat{\theta}, \pi - \frac{1}{n^\delta}\right) \text{ if } \widehat{\theta} \geq \frac{\pi}{2}$$

where $\delta \in \left]0, \frac{2}{7}\right[$ will be fixed latter. Now suppose that the sample $(X_t)_{1 \leq t \leq n}$ has been split into two parts: the first part, for n running from 1 to m_n is devoted to the estimation of θ; the second part is used to the calculation of the tests statistics with $\widehat{\theta}$ in place of θ. We take $m_n = [n/2]$. Obviously, $\widehat{\theta}$ is independent from ε_j for $j > m_n$, and this fact allows us to replace θ by $\widehat{\theta}$ without affecting the limiting distribution of the statistic:

$$\xi_n^r = \left|\frac{e(n)}{\widetilde{e}(n)}\right| \sqrt{\frac{\mathbb{I}_X\left(\widehat{\theta}^* + e(n)^{-1}\right)}{\mathbb{I}_X\left(\widehat{\theta}^* + \widetilde{e}(n)^{-1}\right)}}$$

[7]The assumption about ε_t are not the weakest possible.

[8]We should mention the method proposed by (Newton and Pagano (1983)) for the estimation of periods associated with peaks in the spectrum. But their methodology is not valid for our problem, because it consists in estimating extrema of an autoregressive approximation of the spectrum, and we know that such approximation is not valid under $\mathbf{H}_0$.

where we set $e(n) = \frac{n^\zeta}{C}$ $(\delta < \zeta < \frac{1}{2})$ and $\widetilde{e}(n) = \frac{n^{\widetilde{\zeta}}}{\widetilde{C}}(\delta < \widetilde{\zeta} < \zeta)$, C and $\widetilde{C}$ constant, and $\mathbb{I}_X$ is the periodogram from observations X_{m_n+1} to X_n . We suppose that hypothesis $\mathbf{H}$, with $d \geq 2$ are fulfilled.

Theorem 21 *Under* $\mathbf{H}_0$*, we have* $\xi_n^r \Rightarrow \sqrt{F_{2,2}}$*, whereas under* $\mathbf{H}_a$*,* ξ_n^r *diverge to* $+\infty$

Proof: We use the notations of theorem 19. Set $\theta_n = \widehat{\theta}^* + e(n)^{-1}$ and $\widetilde{\theta}_n = \widehat{\theta}^* + \widetilde{e}(n)^{-1}$. The martingale difference array is now $u_{n,k} = x_{n,k}\varepsilon_k$ with:

$$x_{n,i} = \frac{a\cos(\theta_n k) + b\sin(\theta_n k) + c\cos\left(\widetilde{\theta}_n k\right) + d\sin\left(\widetilde{\theta}_n k\right)}{\sigma\sqrt{\sum_{k=m_n+1}^{n} E\left[a\cos(\theta_n k) + b\sin(\theta_n k) + c\cos\left(\widetilde{\theta}_n k\right) + d\sin\left(\widetilde{\theta}_n k\right)\right]^2}}$$

Let $y_{n,k} = a\cos(\theta_n k) + b\sin(\theta_n k) + c\cos\left(\widetilde{\theta}_n k\right) + d\sin\left(\widetilde{\theta}_n k\right)$

We must prove that $\sum_{k=m_n+1}^{n} u_{n,k}^2 \xrightarrow{P} 1$. Let us show that

$$\sum_{k=m_n+1}^{n} E\left(y_{n,k}^2\right) \sim \frac{a^2+b^2+c^2+d^2}{4} n \tag{40}$$

We write $y_{n,k}^2 = w_{n,k} + z_{n,k}$

with: $w_{n,k} = [a\cos(\theta_n k) + b\sin(\theta_n k)]^2 + \left[c\cos\left(\widetilde{\theta}_n k\right) + d\sin\left(\widetilde{\theta}_n k\right)\right]^2$

and $z_{n,k} = 2\left[a\cos(\theta_n k) + b\sin(\theta_n k)\right]\left[c\cos\left(\widetilde{\theta}_n k\right) + d\sin\left(\widetilde{\theta}_n k\right)\right]$

$$\begin{aligned} E\sum_{k=1}^{n} w_{n,k} = \ & (a^2-b^2)\left\{\frac{n}{4} - \frac{1}{2}E\cos\left[\left(\frac{3n}{2}-1\right)\theta_n\right]\frac{\sin\left(\frac{n}{2}\theta_n\right)}{\sin(\theta_n)}\right\} + \frac{n}{2}b^2 \\ & + (c^2-d^2)\left\{\frac{n}{4} - \frac{1}{2}E\cos\left[\frac{3n}{2} - 1\widetilde{\theta}_n\right]\frac{\sin\left(\frac{n}{2}\widetilde{\theta}_n\right)}{\sin\left(\widetilde{\theta}_n\right)}\right\} + \frac{n}{2}d^2 \\ & + abE\sin\left[\left(\frac{3n}{2}-1\right)\theta_n\right]\frac{\sin(n\theta_n)}{\sin(\theta_n)} + cdE\sin\left[\left(\frac{3n}{2}-1\right)\widetilde{\theta}_n\right]\frac{\sin\left(\frac{n}{2}\widetilde{\theta}_n\right)}{\sin\left(\widetilde{\theta}_n\right)} \end{aligned}$$

But $e(n)^{-1} = o\left(n^{-\delta}\right)$, so $\frac{1}{2}n^{-\delta} \leq \widehat{\theta}^* + e(n)^{-1} \leq \pi - \frac{1}{2}n^{-\delta}$ for n large enough, so we get:

$$\left|\sin\left(\widehat{\theta}^* + e(n)^{-1}\right)\right| \geq \sin\frac{1}{2}n^{-\delta} \geq Cn^{-\delta}$$

and then $\left|\cos\left[(n+1)\theta_n\right]\frac{\sin(n\theta_n)}{\sin(\theta_n)}\right| \leq C'n^\delta$. Now,

$$E\left|\frac{1}{n}\sum_{k=1}^{n} w_{n,k} - \frac{1}{4}\left(a^2+b^2+c^2-d^2\right)\right| = O\left(n^{\delta-1}\right)$$

It yields $E\left|\frac{1}{n}\sum_{k=1}^{n} w_{n,k}\right| \to \frac{1}{2}\left(a^2+b^2+c^2-d^2\right)$

Let's examine a generic term of $\frac{1}{n}\sum_{k=1}^{n} z_{n,k}$

$\frac{1}{n}E\left|\sum_{k=0}^{n-1}\cos(\theta_n k)\cos\left(\widetilde{\theta}_n k\right)\right| \leq \frac{1}{\left|\sin\left(\frac{[\theta_n+\widetilde{\theta}_n]}{2}\right)\right|} + \frac{1}{\left|\sin\left(\frac{[\theta_n-\widetilde{\theta}_n]}{2}\right)\right|}$

The second term has been treated in the proof of theorem 19: indeed, it does not depend on the value of $\widehat{\theta}^*$. As for the first term we use the same argument as before because:
$e(n)^{-1}+\widetilde{e}(n)^{-1}=o\left(n^{-\delta}\right)$, so $\frac{1}{2}n^{-\delta}\leq\frac{\left[\theta_n+\widetilde{\theta}_n\right]}{2}\leq\pi-\frac{1}{2}n^{-\delta}$ for n large enough. Finally, we obtain:

$$\frac{1}{n}E\left|\sum_{k=1}^{n}z_{n,k}\right|=O\left(n^{\delta-1}\right)$$

and (40) is proved.

Now, $\sum_{k=m_n+1}^{n}u_{n,k}^2-1=\frac{\sum_{k=m_n+1}^{n}\left(y_{n,k}^2\varepsilon_k^2-\sigma^2Ey_{n,k}^2\right)}{\sigma^2\sum_{k=m_n+1}^{n}Ey_{n,k}^2}$. Define $Z_{n,k}=y_{n,k}^2\varepsilon_k^2-\sigma^2Ey_{n,k}^2$

$$\begin{aligned}E\left(\sum_{k=m_n+1}^{n}Z_{n,k}^2\left|\mathcal{F}_{m_n}\right.\right)&=\sum_{k=m_n+1}^{n}E\left(Z_{n,k}^2\left|\mathcal{F}_{m_n}\right.\right)\\&=\sum_{k=m_n+1}^{n}\left(y_{n,k}^4E\varepsilon_k^4-\sigma^4\left(Ey_{n,k}^2\right)^2\right)\end{aligned}$$

Note that $|y_{n,k}|\leq1$, so we get $E\left(\sum_{k=m_n+1}^{n}Z_{n,k}^2\left|\mathcal{F}_{m_n}\right.\right)=O\left(n\right)$, and by the law of iterated expectations, $E\left(\sum_{k=m_n+1}^{n}Z_{n,k}^2\right)=O\left(n\right)$. Finally,

$$\mathbb{E}\left(\sum_{k=m_n+1}^{n}u_{n,k}^2-1\right)^2=O\left(\frac{1}{n}\right)$$

It yields $\sum_{k=m_n+1}^{n}u_{n,k}^2\xrightarrow{P}1$. Note now that $|u_{n,k}|\leq\frac{(|a|+|b|+|c|+|d|)}{\sigma\sqrt{\sum_{k=m_n+1}^{n}Ey_{n,k}^2}}|\varepsilon_k|\leq\frac{C}{\sqrt{n}}|\varepsilon_k|$ and $\max_{1\leq k\leq n}|u_{n,k}|=o_p\left(1\right).$

We finally get:

$$\frac{1}{\sqrt{n}}\begin{pmatrix}\sum_{k=m_n+1}^{n}\cos\left(\theta_nk\right)\\\sum_{k=m_n+1}^{n}\sin\left(\theta_nk\right)\\\sum_{k=m_n+1}^{n}\cos\left(\widetilde{\theta}_nk\right)\\\sum_{k=m_n+1}^{n}\cos\left(\widetilde{\theta}_nk\right)\end{pmatrix}\Rightarrow\frac{\sigma}{2}\begin{pmatrix}\mathbb{N}_1\\\mathbb{N}_2\\\mathbb{N}_3\\\mathbb{N}_4\end{pmatrix}$$

with $\mathbb{N}_j$ i.i.d standard normal. Then:

$$\begin{pmatrix}\mathbb{I}_\varepsilon\left(\theta_n\right)\\\mathbb{I}_\varepsilon\left(\widetilde{\theta}_n\right)\end{pmatrix}\Rightarrow\frac{\sigma^2}{4}\begin{pmatrix}\chi_2^1\\\chi_2^2\end{pmatrix}$$

Now, from (11) with $t=1:\mathbb{E}\left|e(n)\mathbb{I}_X\left(\theta_n\right)-\frac{2\pi}{\sigma^2}\mathbb{I}_\varepsilon\left(\theta_n\right)e(n)f\left(\theta_n\right)\right|=o\left(1\right)$ and $e(n)f\left(\theta_n\right)=e(n)f\left(\widehat{\theta}^*+e(n)^{-1}\right)$. $f^{(2)}$ is continuous on $\mathbb{R}$. So we get, for some constant $C>0$: $\left|f\left(\widehat{\theta}^*+e(n)^{-1}\right)-f\left(\widehat{\theta}^*\right)-e(n)^{-1}f'\left(\widehat{\theta}^*\right)\right|\leq Ce(n)^{-2}$ and:
$\left|e(n)f\left(\theta_n\right)-e(n)f\left(\widehat{\theta}^*\right)-f'\left(\widehat{\theta}^*\right)\right|\leq Ce(n)^{-1}$

Now, under $\mathbf{H_0}$, $\left|f\left(\widehat{\theta}^*\right)\right|\leq C'\left|\widehat{\theta}^*-\theta\right|$, so we get:

$$\left|e(n)f\left(\theta_n\right)-f'\left(\widehat{\theta}^*\right)\right|=O\left(e(n)^{-1}+e(n)\left|\widehat{\theta}^*-\theta\right|\right)$$

We see now that:
$e(n)\left|\widehat{\theta}^*-\theta\right|=2e(n)\left|\widehat{\theta}^*-\theta\right|+C\times e(n)\times1\left(\widehat{\theta}<\frac{1}{n^\delta}\right)+C'\times e(n)\times1\left(\widehat{\theta}>\pi-\frac{1}{n^\delta}\right)$

By $\widehat{\theta}-\theta=o_p\left(n^{-\frac{2}{7}}\right)$, we get $e(n)\left|\widehat{\theta}^{*}-\theta\right|=o_p(1)$ and:

$$e(n)f(\theta_n)\xrightarrow{P}f^{'}(\theta)$$

by continuity of $f^{'}$. The treatment of quantities involving $\widetilde{\theta}_n$ is similar. The desired result follows immediately.

Under $\mathbf{H}_a$, $f(\theta_n)\xrightarrow{P}f(\theta)$, then $\sqrt{\frac{\mathbb{I}_X\left(\widehat{\theta}^{*}+e(n)^{-1}\right)}{\mathbb{I}_X\left(\widehat{\theta}^{*}+\widetilde{e}(n)^{-1}\right)}}$ converges in distribution. Since $\left|\frac{e(n)}{\widetilde{e}(n)}\right|\to+\infty$, ξ_n^r diverges.

■

7 Conclusion

This paper presents results related to the behavior of the finite Fourier transform of a sample extracted from a stationary time series. Building upon functional limit theorems, one can easily derive statistics of the hypothesis that the spectral density vanishes for some given frequency. The interest of this method is three-fold. The calculation of all statistics is quite simple with standard asymptotic distribution, the case of unknown frequency can be dealt with, and lastly the method leaves the possibility to handle series which are only locally regular: this point is currently under investigation, and partial results indicate for instance that we may get an estimate of the fractional degree of differencing for long memory processes. The drawbacks of the approach developed in this paper are classical when dealing with non-parametric methods: estimators converge rather slowly, and tests are sub-optimal once the model has been plugged into a parametric framework.

8 Appendix

8.1 Proof of lemma 1

Let us consider the complex stationary process $Y_t = e^{-i\theta t} X_t$. Y_t is PND, and its spectral density is $f_Y(\omega) = f(\omega + \theta)$. Let $\varepsilon > 0$ such as $]\theta - \varepsilon, \theta + \varepsilon[\subset V(\theta)$. we have:

$$\mathbb{I}(X,\theta) = \int_{-\pi}^{\pi} \frac{f_Y(x)}{\sin^2(x/2)} dx \leq \int_{[-\pi,\pi]\setminus[\theta-\varepsilon,\theta+\varepsilon]} \frac{f(x)}{\sin^2([x-\theta]/2)} dx + C \int_{\theta-\varepsilon}^{\theta+\varepsilon} \frac{|x-\theta|^{1+\alpha}}{\sin^2([x-\theta]/2)} dx$$

Let $m = \inf\left\{\sin^2([x-\theta]/2), x \in [-\pi,\pi]\setminus[\theta-\varepsilon, \theta-\varepsilon]\right\} > 0$. From $|\sin y| \geq \frac{2}{\pi}|y|$ for all $|y| \leq \varepsilon$:

$$\mathbb{I}(X,\theta) \leq m^{-1} \int_{-\pi}^{\pi} f(x)\, dx + C' \int_{\theta-\varepsilon}^{\theta+\varepsilon} |x-\theta|^{\alpha-1} dx < \infty$$

because f is integrable and $\alpha > 0$.
Thus the function $\varphi(\omega) = (1 - e^{-i\omega})^{-1}$ belongs to $\mathbb{L}_2([-\pi,\pi], f_Y dy)$. From the spectral representation[9] of Y_t, we define a stationary process by:

$$\widetilde{Y}_t = \int_{-\pi}^{\pi} e^{it\omega} \varphi(\omega)\, dZ_Y(\omega)$$

moreover, this process belongs to H_Y and verifies:

$$Y_t = \widetilde{Y}_t - \widetilde{Y}_{t-1} \tag{41}$$

We proceed to show that $\widetilde{Y}_t \in H_Y(t) = \overline{\{Y_j, j \leq t\}}$. Classically, we consider:

$$\widetilde{Y}_{t,m} = \sum_{j=0}^{m} \left(1 - \frac{j}{m}\right) Y_{t-j} \in H_Y(t)$$

It is clear that:
$\widetilde{Y}_{t,m} - \widetilde{Y}_t = \frac{1}{m}\sum_{j=1}^{m} \widetilde{Y}_{t-j}$ thus $\mathbb{E}\left(\widetilde{Y}_{t,m} - \widetilde{Y}_t\right)^2 = \frac{1}{m^2}\int_{-\pi}^{\pi} \frac{\sin^2(m\omega/2)}{\sin^2(\omega/2)} |\varphi(\omega)|^2 f(\omega+\theta)\, d\omega$
Let $H(\omega) = |\varphi(\omega)|^2 f(\omega+\theta)$. Pour $|\omega| < \varepsilon$, $H(\omega) \leq C|\omega|^{\alpha-1}$
From a well known argument, and the integrability of H we get:

$$\int_{-\pi}^{\pi} \frac{\sin^2(m\omega/2)}{\sin^2(\omega/2)} H(\omega)\, d\omega = \int_{-\varepsilon}^{\varepsilon} \frac{\sin^2(m\omega/2)}{\omega^2} H(\omega)\, d\omega + O(1)$$

$$\begin{aligned} \int_{-\varepsilon}^{\varepsilon} \frac{\sin^2(m\omega/2)}{\omega^2} H(\omega)\, d\omega &\leq 2\int_0^{\varepsilon} \frac{\sin^2(m\omega/2)}{\omega^{3-\alpha}} d\omega = Cm^{2-\alpha} \int_0^{m\varepsilon} \frac{\sin^2(x)}{x^{3-\alpha}} dx \\ &\leq Cm^{2-\alpha}\left[C' + \int_{\varepsilon}^{m\varepsilon} \frac{dx}{x^{3-\alpha}}\right] \end{aligned}$$

[9] $Z_Y(\omega)$ is the process with orthogonal increments (P.O.I) associated to Y_t.

But $\int_{\varepsilon}^{m\varepsilon} \frac{dx}{x^{3-\alpha}}$ $= O\left(m^{\alpha-2}\right)$ if $\alpha > 2$
$= O(1)$ if $\alpha < 2$
$= O(\log m)$ if $\alpha = 2$
Thus, in all cases $\int_{-\pi}^{\pi} \frac{\sin^2(m\omega/2)}{\sin^2(\omega/2)} H(\omega)\, d\omega = O\left(m^{2-\alpha} + \log m\right)$, so:

$$\mathbb{E}\left(\widetilde{Y}_{t,m} - \widetilde{Y}_t\right)^2 = O\left(m^{-\alpha} + \frac{\log m}{m^2}\right) = o(1)$$

and the result follows.
Now, write $\widehat{Y}_t = e^{i\theta t}\widetilde{Y}_t$. We have $X_t = e^{i\theta t} Y_t$ which can be written:

$$\begin{array}{l} X_t = \left(1 - e^{i\theta} B\right) \widehat{Y}_t \\ \widehat{Y}_t \in H_Y(t) \subset H_X(t) \end{array} \tag{42}$$

Suppose that $\theta \notin \{0, \pi\}$. Let us consider the stationary process $Z_t = e^{i\theta t}\widehat{Y}_t$ with spectral density. $f_Z(t) = \frac{f(\omega-\theta)}{4\sin^2\theta}$. The argument used previously gives
$\int_{-\pi}^{\pi} \frac{f_Z(x)}{\sin^2(x/2)} dx < \infty$ and we obtain a result analogous to (**??**):
$\exists\ \widetilde{Z}_t$ stationary process such as $Z_t = \widetilde{Z}_t - \widetilde{Z}_{t-1}$ and $\widetilde{Z}_t \in H_Z(t) \subset H_{\widehat{Y}}(t) \subset H_X(t)$.
Next, we get:

$$\widehat{Y}_t = e^{-i\theta t} Z_t = u_t - e^{-i\theta} u_{t-1}$$

with $u_t = e^{-i\theta t}\widetilde{Z}_t$. From (**??**) we obtain:

$$X_t = \left(1 - e^{i\theta} B\right)\left(1 - e^{-i\theta} B\right) u_t = \left(1 - 2\cos\theta B + B^2\right) u_t \tag{43}$$

$(u_t)_{t\in\mathbb{Z}}$ is a complex process verifying $u_t \in H_X(t)$. But it is easily seen that u_t is real. Indeed, let Z_u and Z_X be the POI associated with u_t and X_t respectively. Then (43) yields:
$dZ_u(\omega) = \left(1 - 2\cos\theta e^{i\omega} + e^{2i\omega}\right)^{-1} dZ_X(\omega)$ and $\overline{dZ_u(\omega)} = dZ_u(-\omega)$ which means that u_t is real.

Now, (42) when $\theta \in \{0, \pi\}$, and (43) when $\theta \notin \{0, \pi\}$, yields the desired result.
Let's now turn to the unicity of u_t. If v_t is another solution, let $d_t = u_t - v_t$.
$\left(1 - 2\cos\theta B + B^2\right) d_t = 0$, thus $\exists A$ stochastic variable such as $d_t = Ae^{i\theta t}$.
Both u_t and v_t belong to $H_X(t)$, and $d_t \in H_X(t)$, so:

$$A = e^{i\theta t} d_t \in \bigcap_{t=-\infty}^{+\infty} H_X(t) = \{0\}$$

because X_t is PND.
■

8.2 Proof of lemma 3

Firstly, we remark that $\log f(\omega) = \frac{1}{2}a_0 + \sum_{s=1}^{\infty} a_s e^{is\omega}$, the $(a_s)_{s\geq 0}$ being the Fourier development of $\log \Psi(\omega) = \log\left(\sum_{j=0}^{\infty} \Psi_j e^{ij\omega}\right)$. Notice that the convergence involved here is both pointwise and in quadratic mean because $\log f$ is continuous. From the hypothesis, this function is C^p, so we get from standard results

$$a_h = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{iht}\log f(t)\,dt = -\frac{(-1)^p}{2\pi h^p}\int_{-\pi}^{\pi} e^{iht}\left(\log f(t)\right)^{(p)} dt = .o(h^{-p})$$

Suppose now for example that $f^{(p)}$ admits no derivative at α and π. By integrating by part on $[-\pi,-\alpha]$, $[-\alpha,\alpha]$, $[\alpha,\pi]$ we obtain the bound $a_h = O\left(h^{-p-1}\right)$.
Next, $\Psi'(z) = \Psi(z)\left(\sum_{s=0}^{\infty} a_s z^s\right)$ and $\Psi_n = \frac{1}{n}\sum_{k=1}^{n} k a_k \Psi_{n-k}$ pour $n \geq 1$. We set now, for $1 < \alpha < p$ $C(\alpha, m) = \sup\{0 \leq j \leq m, (1+j^\alpha)|\Psi_j|\}$. $C(\alpha,m)$ is increasing with m. Let $x \in \left]\frac{\alpha}{p}, 1\right[$.

$$\begin{aligned}(1+n^\alpha)|\Psi_n| &\leq \tfrac{C(\alpha,n-1)(1+n^\alpha)}{n}\textstyle\sum_{k=1}^{n}\tfrac{k|a_k|}{(1+(n-k)^p)}\\ &\leq \tfrac{MC(p,n-1)}{n}\textstyle\sum_{k=1}^{n}\tfrac{1}{k^p(1+(n-k)^\alpha)}\end{aligned}$$

with $M = \sup\left(h^{p+1}|a_h|\right)$. Hence:

$$\begin{aligned}(1+n^\alpha)|\Psi_n| &\leq \tfrac{MC(\alpha,n-1)(1+n^\alpha)}{n}\left[\textstyle\sum_{k=1}^{[n^x]}\tfrac{1}{k^p(1+(n-k)^\alpha)} + \textstyle\sum_{k=[n^x]+1}^{n}\tfrac{1}{k^p(1+(n-k)^\alpha)}\right]\\ &\leq \tfrac{MC(\alpha,n-1)(1+n^\alpha)}{n}\left[\tfrac{[n^x]}{(1+(n-[n^x])^\alpha)} + \tfrac{n-[n^x]}{[n^x]^p}\right]\\ &\leq MC(\alpha,n-1)\left[\tfrac{(1+n^\alpha)n^{x-\alpha-1}}{(1-([n^x]/n)^\alpha)} + \tfrac{(1+n^\alpha)}{[n^x]^p}\right]\end{aligned}$$

For some integer N_0, the expression between brackets is an $o(1)$ for all $n \geq N_0$: $(1+n^\alpha)|\Psi_n| \leq MC(\alpha,n-1)\times\frac{1}{M} = C(\alpha,n-1)$, and $C(\alpha,n) = C(\alpha,n-1)$ for $n \geq N_0$. It yields $\Psi_n = O\left(n^{-\alpha}\right)$ and the lemma follows for all $d \in \left]0,\alpha-1\right[$.
■

8.3 Proof of theorem 7

Let $(a,b) \in \mathbb{R}^2$ be fixed. We define $u_{n,k} = x_{n,k}\varepsilon_k$ with:

$$x_{n,k} = \frac{a\cos(\theta_n k) + b\sin(\theta_n k)}{\sigma\sqrt{\sum_{k=1}^{n}\left(a\cos(\theta_n k) + b\sin(\theta_n k)\right)^2}} \text{ and } \sigma_{n,k}^2 = \mathrm{Var}(u_{n,k})$$

Let $v_{n,k} = \sum_{k=1}^{k} u_{nk}$, and $Y_n(t) = v_{n,[nt]} + (nt - [nt])\,u_{n,[nt]+1}$.
Now we show that $Y_n(t)$ converge weakly to a standard Brownian motion. Indeed, $(u_{n,k})$ is a martingale difference array adapted to the filtration $(\mathbb{F}_k)$ and $\sum_{k=1}^{n}\mathrm{var}(u_{n,k}) = 1$. We proceed to the following steps:

- $\underline{\sum_{k=1}^{n} u_{n,k}^2 \xrightarrow{P} 1}$

  $\sum_{k=1}^{n} u_{n,k}^2 - 1 = \frac{\sum_{k=1}^{n}(a\cos(\theta_n k)+b\sin(\theta_n k))^2\left(\varepsilon_k^2-\sigma^2\right)}{\sigma^2\sum_{k=1}^{n}(a\cos(\theta_n k)+b\sin(\theta_n k))^2} = \frac{\sum_{k=1}^{n} y_{n,k}^2 Z_k}{\sigma^2\sum_{k=1}^{n} y_{n,k}^2}$ with $Z_k = \frac{\epsilon_k^2-\sigma^2}{\sigma^2}$, $y_{n,k} = a\cos(\theta_n k) + b\sin(\theta_n k)$. From the trigonometrical identities:

$$\sum_{k=1}^{n}\cos^2(\theta_n k) = \frac{n}{2} - \frac{1}{2}\cos\left[(n+1)\theta_n\right]\frac{\sin(n\theta_n)}{\sin(\theta_n)}$$

$$\sum_{k=1}^{n} \sin(\theta_n k) \cos(\theta_n k) = \frac{1}{2} \sin[(n+1)\theta_n] \frac{\sin(n\theta_n)}{\sin(\theta_n)}$$

we get $\left|\sum_{k=1}^{n} y_{n,k}^2 - (a^2 - b^2)\frac{n}{2} + nb^2\right| \leq \frac{2}{|\sin(\theta_n)|}$, and:

$$\left|\frac{1}{n}\sum_{k=1}^{n} y_{n,k}^2 - \frac{(a^2+b^2)}{2}\right| \leq \frac{2}{n|\sin(\theta_n)|}$$

When $\theta \notin \{0, \pi\}$. From the convergence of θ_n to $\theta \neq 0$ and π, the term in the right hand side of the inequality is $O(n^{-1})$ and $\sum_{k=1}^{n} y_{n,k}^2 \sim \frac{a^2+b^2}{2} n$ .When $\theta = \pi$, $n\sin(\theta_n) \sim -n(\theta_n - \pi) \to +\infty$, and for $\theta = 0$, $n\sin(\theta_n) \sim n\theta_n \to +\infty$ by assumption. Thus, in all cases:

$$\sum_{k=1}^{n} y_{n,k}^2 \sim \frac{a^2+b^2}{2} n$$

Now, we get:
$\varepsilon_k^2 - \sigma_k^2 = \varepsilon_k^2 - \sigma^2 - \sum_{j=1}^{\infty} c_j \left(\varepsilon_{k-j}^2 - \sigma^2\right)$. Define $\eta_t = \varepsilon_t^2 - \sigma_t^2$ and $\Psi(z) = 1 - \sum_{j=1}^{\infty} c_j z^j$

$$\eta_k = \Psi(B)\left(\varepsilon_k^2 - \sigma^2\right)$$

Because Ψ is square-summable, and doesn't vanish in the unit disk:

$$\varepsilon_k^2 - \sigma^2 = \Psi^{-1}(B)\,\eta_k$$

We set $\Psi^{-1}(z) = \sum_{j=0}^{\infty} d_j z^j$ for a sequence (d_j) satisfying $\sum_{j=0}^{\infty} |d_j| < \infty$. By assumption, $(\eta_k, \mathbb{F}_k)$ is a m.d.s. It is easily seen that its moments up to order 2 are bounded: $\sigma_t^2 = \sigma^2 + \sum_{k=1}^{\infty} c_k \left(\varepsilon_{t-k}^2 - \sigma^2\right)$, and:

$$\left\|\sigma_t^2\right\|_2 \leq \sigma^2 + \sum_{k=1}^{\infty} |c_k| \left(\|\varepsilon_{t-k}\|_2^2 + \sigma^2\right) < \infty$$

$$\sup \mathbb{E}\left(\sigma_t^4\right) \leq \left[\sigma^2\left(1 + \sum_{k=1}^{\infty}|c_k|\right) + \left(\sum_{k=1}^{\infty}|c_k|\right)\sup \mathbb{E}\left(\varepsilon_t^4\right)\right]^4 \text{ and } \sup \mathbb{E}\left(\eta_t^2\right) < \infty$$

Now, let $\Delta_n = \sum_{k=1}^{n} y_{n,k}^2 Z_k$.

$$\begin{aligned}
\Delta_n &= \textstyle\sum_{k=1}^{n} y_{n,k}^2 \left[\sum_{j=0}^{\infty} d_j \eta_{k-j}\right] \\
\mathbb{E}\Delta_n^2 &= \textstyle\sum_{u,v=1}^{n} y_{n,u}^2 y_{n,v}^2 \left(\sum_{j-k=u-v} d_j d_k \mathbb{E}\eta_{u-j}^2\right) \\
&= \textstyle\sum_{u,v=1}^{n} y_{n,u}^2 y_{n,v}^2 \left(\sum_{j=\max(u-v,0)}^{\infty} d_j d_{j-u+v} \mathbb{E}\eta_{u-j}^2\right)
\end{aligned}$$

But $y_{n,u}^2 \leq C$ and $\mathbb{E}\eta_j^2 \leq C$, so:

$$\mathbb{E}\Delta_n^2 \leq \sum_{u=1}^{n} u\left(\sum_{j=u}^{\infty} |d_j d_{j-u}|\right) \leq n\left(\sum_{j=0}^{\infty} |d_j|\right)^2$$

and finally we get:

$$\mathbb{E}\left(\sum_{k=1}^{n} u_{n,k}^2 - 1\right)^2 \leq C\frac{\mathbb{E}\Delta_n^2}{n^2} = O\left(\frac{1}{n}\right)$$

the result follows from Chebychev's inegality.

- $\underline{\lim_{n\longrightarrow\infty} \sum_{k=1}^{[nt]} \sigma_{n,k}^2 = t \text{ for all } t \in [0,1]}$.

  From $\sigma_{n,k}^2 = \frac{y_{n,k}^2}{\sum_{k=1}^{n} y_{n,k}^2}$, the result is obvious since $\sum_{k=1}^{[nt]} y_{n,k}^2 \sim \frac{a^2+b^2}{2}[nt]$.

- $\underline{\max_{1\leq k\leq n} |u_{n,k}| \underset{n\to\infty}{\to} \text{ in probability.}}$

  Indeed, $|u_{n,k}| \leq \frac{(|a|+|b|)}{\sigma\sqrt{\sum_{k=1}^{n} y_{n,k}^2}}\varepsilon_k \leq \frac{C}{\sqrt{n}}\varepsilon_k$. Classically, for all $\delta > 0$ we have:

$$\begin{aligned} P\left(\max_{1\leq k\leq n} |u_{n,k}| > \delta\right) &\leq P\left(\max_{1\leq k\leq n} \left|\tfrac{\varepsilon_k}{\sqrt{n}}\right| > \tfrac{\delta}{C}\right) \\ &\leq P\left(\tfrac{1}{n}\sum_{k=1}^{n} \varepsilon_k^2 \mathbb{I}\left[\varepsilon_k^2 > \tfrac{\delta^2}{C^2}n\right] > \tfrac{\delta^2}{C^2}\right) = P\left(\mathbb{J}_n > \tfrac{\delta^2}{C^2}\right) \end{aligned}$$

with $\mathbb{J}_n = \frac{1}{n}\sum_{k=1}^{n} \varepsilon_k^2 \mathbb{I}\left[\varepsilon_k^2 > \frac{\delta^2}{C^2}n\right]$. An application of Cauchy-Schwartz's and Chebychev's inegality yields:

$$\begin{aligned} \mathbb{E}(\mathbb{J}_n) &\leq \tfrac{1}{n}\sum_{k=1}^{n} \sqrt{\mathbb{E}(\varepsilon_k^4)}\sqrt{P\left[\varepsilon_k^2 > \tfrac{\delta^2}{C^2}n\right]} \\ &\leq \tfrac{C}{n}\sum_{k=1}^{n} \sqrt{\tfrac{\mathbb{E}(\varepsilon_k^4)C^2}{\delta^2 n}} \leq \tfrac{C'}{\sqrt{n}} \end{aligned}$$

and then $\mathbb{J}_n \xrightarrow{P} 0$.
The hypotheses for the functional central limit theorem are fulfilled (Davidson (1994)), so we get:

$$Y_n(t) \underset{n\to\infty}{\Rightarrow} \mathbb{B}(t) \text{ with } \mathbb{B}(t) \text{ standard Brownian motion} \tag{44}$$

(42) can be written $\left(\sum_{k=1}^{n} y_{n,k}^2\right)\left(\frac{a^2+b^2}{2}n\right)^{-1/2} Y_n(t) \underset{n\to\infty}{\Rightarrow} \mathbb{B}(t)$,or:
$a\sqrt{\frac{2}{n}}\left[\sum_{k=1}^{[nt]} \varepsilon_k \cos(k\theta_n) + (nt-[nt])\cos(([nt]+1)\theta_n)\varepsilon_{[nt]+1}\right] +$
$b\sqrt{\frac{2}{n}}\left[\sum_{k=1}^{[nt]} \varepsilon_k \sin(k\theta_n) + (nt-[nt])\sin([nt]+1)\theta_n\varepsilon_{[nt]+1}\right] \Rightarrow \sigma\sqrt{a^2+b^2}\mathbb{B}(t)$
In a more compact form:

$$(a,b)\,T_n(t,\theta_n) \Rightarrow \sigma\sqrt{a^2+b^2}\mathbb{B}(t) \equiv \sigma\{a\mathbb{B}_1(t) + b\mathbb{B}_2(t)\}$$

where $\mathbb{B}_1(t)$ and $\mathbb{B}_2(t)$ are two independent Brownian motions, the equality defined in the sense of distributions over the space $C(0,1)$. From the Cramer-Wold device, it yields the convergence of $T_n(t,\theta_n)$ to $(\mathbb{B}_1(t),\mathbb{B}_2(t))'$.
■

8.4 Proof of lemma 10

The proof follows quite classical arguments. We begin with the following function:

$$W_n(\omega, t) = \sqrt{2}\mathbb{J}_X(\omega, t) = C_n(\omega, t) + iS_n(\omega, t), (\omega, t) \in [-\pi, \pi] \times [0, 1]$$

We saw that $W_n(\theta, t) + \sqrt{2}\widehat{T_n(t)} \Rightarrow \sigma C_X\left(e^{-i\theta}\right)\mathbb{W}_c(t)$; the convergence is achieved for all $\theta \in [0, \pi]$, and the limit may be degenerate in case $f(\theta) = 0$. Moreover:

$$\left(\sqrt{\frac{2}{n}}\sin\theta\right)Y_t = \sin\left[(t+1)\theta\right]C_n\left(\theta, \frac{t}{n}\right) - \cos\left[(t+1)\theta\right]S_n\left(\theta, \frac{t}{n}\right)$$

$$\begin{aligned}\left(\sqrt{\tfrac{2}{n}}\sin\theta\right)\textstyle\sum_{k=1}^n Y_k e^{-ik\omega} &= -\tfrac{1}{2}\left[\textstyle\sum_{k=1}^n\left\{C_n\left(\theta, \tfrac{k}{n}\right) - iS_n\left(\theta, \tfrac{k}{n}\right)\right\}e^{ik(\theta-\omega)}\right]ie^{i\theta} \\ &\quad -\tfrac{1}{2}\left[\textstyle\sum_{k=1}^n\left\{C_n\left(\theta, \tfrac{k}{n}\right) + iS_n\left(\theta, \tfrac{k}{n}\right)\right\}e^{-ik(\theta+\omega)}\right]ie^{-i\theta}\end{aligned}$$

or, in complex form:

$$\begin{aligned}\tfrac{2\sqrt{2}\sin(\theta)}{n}\mathbb{J}_Y(\omega) &= -ie^{i\theta}\left[\tfrac{1}{n}\textstyle\sum_{k=1}^n W_n\left(-\theta, \tfrac{k}{n}\right)e^{ik(\theta-\omega)}\right] \\ &\quad -ie^{-i\theta}\left[\tfrac{1}{n}\textstyle\sum_{k=1}^n W_n\left(\theta, \tfrac{k}{n}\right)e^{-ik(\theta+\omega)}\right]\end{aligned}$$

From the proposition 8 of Jeganathan (1991) we have

$$\frac{1}{n}\sum_{k=1}^n W_n\left(\theta, \frac{k}{n}\right)e^{-ik(\theta+\omega)} = o_p(1)$$

for all $\omega \in [0, \pi]$, and

$$\frac{1}{n}\sum_{k=1}^n W_n\left(-\theta, \frac{k}{n}\right)e^{ik(\theta-\omega)} = o_p(1)$$

for all $\omega \in [0, \pi] \setminus \{\theta\}$. It is easy to verify that we have also:

$$b_n(j) = \frac{1}{n}\sum_{k=1}^n W_n\left(\theta, \frac{k}{n}\right)e^{-ik\left(\theta + \frac{2\pi j}{n}\right)} = o_p(1)$$

uniformly in j for some integer j being fixed. Moreover:

$$\frac{1}{n}\sum_{k=1}^n W_n\left(\theta, \frac{k}{n}\right)e^{ik\frac{2\pi j}{n}} + \frac{\sqrt{2}}{n}\sum_{k=1}^n \widehat{T}_n\left(\frac{k}{n}\right) \Rightarrow \sigma C\left(e^{-i\theta}\right)\int_0^1 e^{2i\pi jt}\mathbb{W}_c(t)dt$$

But $\left|\frac{\sqrt{2}}{n}\sum_{k=1}^n \widehat{T}_n\left(\frac{k}{n}\right)\right| \leq \sqrt{2}\sup_{t\in[0,1]}\left|\widehat{T_n(t)}\right| \xrightarrow{P} 0$, so we get:

$$\frac{1}{n}\sum_{k=1}^n W_n\left(-\theta, \frac{k}{n}\right)e^{ik\frac{2\pi j}{n}} \Rightarrow \sigma C_X\left(e^{-i\theta}\right)\int_0^1 e^{-2i\pi jt}\overline{\mathbb{W}_c(t)}dt$$

$\overline{\mathbb{W}_c(t)}$ is also a complex standard Brownian motion.

When $\theta=\pi$, $(1+B)Y_t=X_t$ and $\sqrt{\frac{2}{n}}Y_t=\cos(t\pi)\,C_n\left(\pi,\frac{t}{n}\right)$, hence:

$$\sqrt{2}\frac{\mathbb{J}_Y(\omega)}{n}=\frac{1}{n}\sum_{k=1}^{n}C_n\left(\pi,\frac{k}{n}\right)e^{ik(\pi-\omega)}$$

$$\sum_{k=1}^{n}C_n\left(\pi,\frac{k}{n}\right)e^{-ik\frac{2\pi j}{n}}+\widehat{\sqrt{2}T_n(t)}\Rightarrow\sqrt{2}\sigma C_X\left(e^{-i\pi}\right)\int_0^1 e^{-2i\pi jt}\mathbb{W}_r(t)dt.$$

■

8.5 Proof of lemma 11

We define the "theoretical" (i.e. non feasible) estimator of $f_u(\theta)$:

$$\widehat{f(\theta)}=\frac{1}{2\pi}\sum_{j=-m}^{m}\frac{1}{2m+1}\mathbb{I}_u\left(\theta+\frac{2\pi j}{n}\right)$$

We first prove that this estimator is consistent under $\mathbf{H_0}$. We follow Brockwell and Davis (1986) and adapt the proof to the presence of conditional heteroskedasticity By assumption $d\geq 2$, $\sum_{k=1}^{\infty}k\left|b\left(k,\theta\right)\right|<\infty$, so f_u is C^1 and:

$$\mathbb{E}\left|\mathbb{I}_u(\omega)-\frac{2\pi}{\sigma^2}\mathbb{I}_\varepsilon(\omega)\,f_u(\omega)\right|=O\left(\frac{1}{\sqrt{n}}\right)$$

The bound is uniform in ω. It yields:

$$\begin{aligned}\widehat{f(\theta)}&=\sigma^{-2}\textstyle\sum_{j=-m}^{m}\frac{1}{2m+1}\mathbb{I}_\varepsilon\left(\theta+\frac{2\pi j}{n}\right)f_u\left(\theta+\frac{2\pi j}{n}\right)+O_p\left(\frac{1}{\sqrt{n}}\right)\\&=\sigma^{-2}\textstyle\sum_{j=-m}^{m}\frac{1}{2m+1}\mathbb{I}_\varepsilon\left(\theta+\frac{2\pi j}{n}\right)\left[f_u\left(\theta+\frac{2\pi j}{n}\right)-f_u(\theta)\right]\\&\quad+\sigma^{-2}\textstyle\sum_{j=-m}^{m}\frac{1}{2m+1}\mathbb{I}_\varepsilon\left(\theta+\frac{2\pi j}{n}\right)f_u(\theta)+O_p\left(\frac{1}{\sqrt{n}}\right)\\\widehat{f(\theta)}&=A_{1,n}+A_{2,n}f_u(\theta)+O_p\left(\frac{1}{\sqrt{n}}\right)\text{ say.}\end{aligned}$$

f_u being C^1, $\sup\limits_j\left|f_u\left(\theta+\frac{2\pi j}{n}\right)-f_u(\theta)\right|\leq C\frac{m}{n}$, so $\mathbb{E}(A_{1,n})=O\left(\frac{m}{n}\right)$. For the second term, we work with the martingale representation of $\widehat{f(\theta)}$:

$$\begin{aligned}\sigma^2A_{2,n}&=\textstyle\sum_{j=-m}^{m}\frac{1}{2m+1}\mathbb{I}_\varepsilon\left(\theta+\frac{2\pi j}{n}\right)=\sum_{j=-m}^{m}\frac{1}{2m+1}\times\frac{1}{n}\sum_{a,b=1}^{n}\varepsilon_a\varepsilon_b e^{-i(a-b)\left(\theta+\frac{2\pi j}{n}\right)}\\&=\textstyle\frac{1}{(2m+1)n}\sum_{j=-m}^{m}\sum_{1,a<b}^{n}2\varepsilon_a\varepsilon_b\left\{e^{-i(a-b)\left(\theta+\frac{2\pi j}{n}\right)}+e^{i(a-b)\left(\theta+\frac{2\pi j}{n}\right)}\right\}+\frac{1}{n}\sum_{a=1}^{n}\varepsilon_a^2\\&=\textstyle\frac{1}{(2m+1)n}\sum_{j=-m}^{m}\sum_{1,a<b}^{n}2\varepsilon_a\varepsilon_b\cos\left[\left(\theta+\frac{2\pi j}{n}\right)(a-b)\right]+\frac{1}{n}\sum_{a=1}^{n}\varepsilon_a^2\\&=\textstyle\frac{1}{(2m+1)n}\sum_{j=-m}^{m}\left\{\sum_{b=1}^{n}2\varepsilon_b\sum_{a=1}^{b-1}\varepsilon_a\cos\left[\left(\theta+\frac{2\pi j}{n}\right)(a-b)\right]\right\}+\frac{1}{n}\sum_{a=1}^{n}\varepsilon_a^2\end{aligned}$$

We get:

$$A_{2,n}=\underbrace{\frac{2}{(2m+1)\,n}\sum_{b=1}^{n}\varepsilon_b\sum_{a=1}^{b-1}\sum_{j=-m}^{m}\left\{\varepsilon_a\cos\left(\theta+\frac{2\pi j}{n}\right)(a-b)\right\}}_{D_n}+\frac{1}{n}\sum_{a=1}^{n}\varepsilon_a^2$$

Let $\eta_b = \frac{1}{n(2m+1)} \times \varepsilon_b \sum_{a=1}^{b-1} \varepsilon_a \underbrace{\sum_{j=-m}^{m} \left\{ \cos\left[\left(\theta + \frac{2\pi j}{n} \right) (a-b) \right] \right\}}_{V_{a-b}}$; η_t is a m.d.s. adapted to the filtration $\mathbb{F}_t$, so:

$$\mathbb{E}D_n^2 = \frac{C}{n^2 (2m+1)^2} \sum_{b=1}^{n} \mathbb{E}\left(\eta_b^2\right)$$

Thus, by the Cauchy-Schwartz's inequality: $\mathbb{E}\left(\eta_b^2\right) \leq \sqrt{\mathbb{E}\left(\varepsilon_b^4\right)} \sqrt{\mathbb{E}\left(\sum_{a=1}^{b-1} \varepsilon_a V_{a-b}\right)^4}$
We can write:

$$V_{a-b} = \cos\theta\,(a-b)\, V_{c,a} - \sin\theta\,(a-b)\, V_{s,a}$$

with $V_{c,a} = \sum_{j=-m}^{m} \cos\left[\left(\theta + \frac{2\pi j}{n}\right) a\right]$ and $V_{s,a} = \sum_{j=-m}^{m} \sin\left[\left(\theta + \frac{2\pi j}{n}\right) a\right]$.
We get:

$$\mathbb{E}\left(\sum_{a=1}^{b-1} \varepsilon_a V_{a-b}\right)^4 \leq C\mathbb{E}\left(Z_{1,b}^4 + Z_{1,b}^4\right)$$

with $Z_{1,b} = \sum_{a=1}^{b-1} \varepsilon_a V_{c,a}$ and $Z_{2,b} = \sum_{a=1}^{b-1} \varepsilon_a V_{s,a}$, martingale sequences for the filtration $\mathbb{F}_b$; from Burkholder's inequality we have:

$$\mathbb{E}Z_{1,b}^4 \leq C\mathbb{E}\left(\sum_{a=1}^{b} \varepsilon_a^2 V_{c,a}^2\right)^2$$

$$\begin{aligned} \text{and } \mathbb{E}Z_{1,b}^4 \leq C\mathbb{E}\left(\sum_{a=1}^{b} \varepsilon_a^2 V_{c,a}^2\right)^2 &= C\sum_{a=1}^{b}\sum_{h=1}^{b} \mathbb{E}\left(\varepsilon_a^2 \varepsilon_h^2\right) V_{c,a}^2 V_{c,h}^2 \\ &\leq C' \sum_{a=1}^{b}\sum_{h=1}^{b} V_{c,a}^2 V_{c,h}^2 = C' \left(\sum_{a=1}^{b} V_{c,a}^2\right)^2 \end{aligned}$$

by an application of Cauchy-Schwarz's inequality, and the fact that moments of order four of ε_t are bounded. This result can be written as:

$$\|D_b\|_2^2 = O\left(\frac{1}{n^2 m^2}\left\{\sum_{b=1}^{n}\left(\sum_{a=1}^{b} V_{c,a}^2\right) + \left(\sum_{a=1}^{b} V_{s,a}^2\right)\right\}\right)$$

or:

$$\|D_b\|_2^2 = O\left(\frac{1}{nm^2}\left\{\sum_{a=1}^{n} V_{c,a}^2 + \sum_{a=1}^{n} V_{s,a}^2\right\}\right)$$

Observe now that $|V_{c,a}| + |V_{s,a}| \leq 2\left|\Delta_m\left(\frac{2\pi a}{n}\right)\right|$:

$$\|D_n\|_2^2 = O\left(\frac{1}{nm^2}\sum_{a=1}^{n}\left|\Delta_m\left(\frac{2\pi a}{n}\right)\right|^2\right) \tag{45}$$

The following lemma deals with the asymptotic behavior of $\sum_{a=1}^{n}\left|\Delta_m\left(\frac{2\pi a}{n}\right)\right|^2$.

Lemma 22 $\sum_{a=1}^{n}\left|\Delta_m\left(\frac{2\pi a}{n}\right)\right|^2 = O\left(mn\log m\right)$

Proof: $x\left|\Delta_m(x)\right| \leq 2$ for $x \in]0,\pi]$, and similarly:

$$(2\pi - x)\left|\Delta_m(x)\right| \leq 2 \text{ for } x \in [\pi, 2\pi[$$

We decompose $\sum_{a=1}^{n}\left|\Delta_m\left(\frac{2\pi a}{n}\right)\right|^2 = \left(\sum_{a=1}^{n/m} + \sum_{a=n-n/m}^{n} \sum_{a=n/m}^{n/2} + \sum_{a=n/2}^{n-n/m}\right)\left|\Delta_m\left(\frac{2\pi a}{n}\right)\right|^2$
The first two terms are handled by bounding $\left|\Delta_m\left(\frac{2\pi a}{n}\right)\right|^2$ by m^2. For the third term, we have:

$$\sum_{a=n/m}^{n/2}\left|\Delta_m\left(\frac{2\pi a}{n}\right)\right|^2 \leq \sum_{a=n/m}^{n/2}\frac{2n^2}{4\pi^2 a^2} \leq C\sum_{a=n/m}^{n/2}\frac{n}{a} \times m = Cnm\sum_{a=n/m}^{n/2}\frac{1}{a}$$

But $\sum_{a=n/m}^{n/2}\frac{1}{a} \sim \log n - \log\left(\frac{n}{m}\right) = \log m$. The fourth term is treated in a similar way:

$$\sum_{a=n/2}^{n-n/m}\left|\Delta_m\left(\frac{2\pi a}{n}\right)\right|^2 \leq C\sum_{a=n/2}^{n-n/m}\left(1-\frac{k}{n}\right)^{-2} \leq Cnm\sum_{a=n/2}^{n-n/m}\frac{1}{n-a}$$

and $\sum_{a=n/2}^{n-n/m}\frac{1}{n-a} = \sum_{a=n/m}^{n/2}\frac{1}{a}$
■
From (43), we get:

$$\left\|D_n\right\|_2 = O\left(\sqrt{\frac{\log m}{m}}\right)$$

it yields $D_n = o_p(1)$, and, in the same way of the proof given for theorem 7:

$$\frac{1}{n}\sum_{a=1}^{n}\varepsilon_a^2 \xrightarrow{P} \sigma^2$$

Finally:

$$\widehat{f(\theta)} \xrightarrow{P} \sigma^{-2} \times \sigma^2 \times f_u(\theta) = f_u(\theta)$$

We now define $\theta_{j,n} = \theta + \frac{2\pi j}{n}$.

$$\begin{aligned}
\left|\widehat{f^*(\theta)} - \widehat{f(\theta)}\right| &\leq \frac{1}{2\pi}\sum_{j=-m}^{m}\frac{1}{2m+1}\left|\mathbb{J}^*(\theta_{j,n}) - \mathbb{J}(\theta_{j,n})\right| \times \left|\mathbb{J}^*(\theta_{j,n}) + \mathbb{J}(\theta_{j,n})\right| \\
\left\|\widehat{f^*(\theta)} - \widehat{f(\theta)}\right\|_1 &\leq \frac{1}{2\pi}\sum_{j=-m}^{m}\frac{1}{2m+1}\left\|\mathbb{J}^*(\theta_{j,n}) - \mathbb{J}(\theta_{j,n})\right\|_2 \times \\
&\quad \left[\left\|\mathbb{J}^*(\theta_{j,n})\right\|_2 + \left\|\mathbb{J}(\theta_{j,n})\right\|_2\right]
\end{aligned}$$

The term associated to $j = 0$ is $\frac{1}{2m+1}\left(1 - \frac{1}{n^{2\alpha}}\right)\left\|\mathbb{J}(\theta_n)\right\|_2^2 = O\left(\frac{1}{m}\right)$. For the other terms, lemma 9 provides the bound $\frac{1}{j(2m+1)}$.
From $\sum_{j=1}^{m}\frac{1}{j(2m+1)} \leq \frac{\log m}{m}$ we get:

$$\left\|\widehat{f^*(\theta)} - \widehat{f(\theta)}\right\|_1 = O\left(\frac{\log m}{m}\right) \tag{46}$$

and then $\widehat{f^*(\theta)} \xrightarrow{P} f_u(\theta)$. In particular, under $\mathbf{H}_a^{'}$, we just add the additional hypothesis $f_u(\theta) = 0$, and the consistency of $\widehat{f^*(\theta)}$ yields $f^*(\theta) \xrightarrow{P} 0$.

Under $\mathbf{H}_a$: $\widehat{f^*(\theta)} > \frac{1}{2\pi(2m+1)}\mathbb{I}^*(n) = \frac{1}{2\pi(2m+1)}\mathbb{I}_Y(\theta + \frac{2\pi}{n})$ with the notations used previously. We know that $\frac{\mathbb{I}_Y(\theta+\frac{2\pi}{n})}{n^2}$ converge to a non degenerate law $\mathbb{L}_\infty$, say. Let's fix $a > 0$. $P\left(\widehat{f^*(\theta)} < a\right) \leq P\left(\mathbb{I}_Y(\theta + \frac{2\pi}{n}) < 2\pi am\right) \leq P\left(\frac{1}{n^2}\mathbb{I}_Y(\theta + \frac{2\pi}{n}) < \frac{2\pi am}{n^2}\right)$.
But by assumption $\frac{2\pi am}{n^2} \to 0$, and $\mathbb{L}_\infty$ doesn't put any mass at the origin. It yields $P\left(\widehat{f(\theta)} < a\right) \to 0$ and the result follows.
■

8.6 Proof of theorem 12

We begin with the $\mathbf{H}_a^{'}$ scheme:

$$\xi_n^p = \frac{e^4(n)\mathbb{I}_X(\theta_n)}{f_v(\theta)} \times \frac{f_v(\theta)}{e^2(n)\widehat{f_u(\theta)}}$$

with $X_t = \left(1 - 2\cos\theta + B^2\right)^2 v_t$ and $f_v(\theta) \neq 0$, $u_t = \left(1 - 2\cos\theta + B^2\right) v_t$
By a similar argument the one used to define ξ_n^p :

$$\frac{e^4(n)\mathbb{I}_X(\theta_n)}{f_v(\theta)} \Rightarrow \mathbb{L}_\infty \text{ if } e(n) = o\left(n^{1/4}\right) \tag{47}$$

Let $\theta_{k,n} = \theta + \frac{2\pi k}{n}$.
$\mathbb{E}\left(\mathbb{I}_u(\omega)\right) = \sum\limits_{|j|<n} \left(1 - \frac{|j|}{n}\right)\gamma_u(j)e^{-ij\omega}$
$\left|\mathbb{E}\left(\mathbb{I}_u(\omega)\right) - f_u(\omega)\right| \leq \sum\limits_{|j|<n} \frac{|j|}{n}\gamma_u(j) + \sum\limits_{|j|>n} |\gamma_u(j)| = o\left(\frac{1}{n}\right)$ uniformly in ω, because $d \geq 2$ implies $\gamma_u(j) = O\left(j^{-2}\right)$

$$\mathbb{E}\left(\mathbb{I}_u(\omega)\right) = f_u(\omega) + o\left(\frac{1}{n}\right) \text{ uniformly in } \omega$$

$\left|\mathbb{E}\left(\widehat{f}_u(\theta)\right) - f_u(\theta)\right| = o\left(\frac{1}{n}\right) + \frac{1}{2\pi(2m+1)}\sum_{j=-m}^{m} \left|f_u\left(\theta_{j,n}\right) - f_u\left(\theta\right)\right|$
We know already that $\left|f_u\left(\theta_{k,n}\right) - f_u\left(\theta\right)\right| = O\left(\frac{m}{n}\right)$ uniformly in k.
Hence $\mathbb{E}\left(\widehat{f}_u(\theta)\right) = O\left(\frac{m}{n}\right)$ and by the assumptions for $e\left(n\right)$:

$$\mathbb{E}\left(e^2(n)\widehat{f_u(\theta)}\right) = o\left(1\right)$$

It yields $e^2(n)\widehat{f}_u(\theta) \xrightarrow{P} 0$
From (44), $\left\|\widehat{f_u^*(\theta)} - \widehat{f_u(\theta)}\right\|_1 = o\left(\frac{1}{|e(n)|^2}\right)$ so $e^2(n)\widehat{f}_u^*(\theta) \xrightarrow{P} 0$
From $f_v(\theta) > 0$ $\frac{f_v(\theta)}{e^2(n)\widehat{f_u(\theta)}} \to +\infty$ and $\xi_n^p \to +\infty$. The argument for ξ_n^s is identical, with the more stringent restriction $e(n) = o\left(n^{1/8}\right)$ for (45) and $\mathbb{I}_X(\theta_n)$ replaced by $\sup\limits_{t\in[0,1]} \mathbb{I}_X\left(\theta_n, t\right)$. This concludes the proof of point ii).

We turn now to the proof of point i). Phillips (1991) has already studied the behavior of spectral density estimator at zero frequency for an integrated process in a somewhat different context. For the sake of clarity and completeness, we provide a different proof of the result, valid for all θ. We build upon lemma (10), and assume $\theta < \pi$ (the case $\theta = \pi$ is similar). We set $\omega_j = \theta + \frac{2\pi j}{n}$.

$\frac{8\sin^2(\theta)}{n^2}\mathbb{I}_Y(\omega_j) = \left|\frac{1}{n}\sum_{k=1}^n W_n\left(-\theta,\frac{k}{n}\right)e^{-ik\frac{2\pi j}{n}}\right|^2 + |b_{n,j}|^2 + \frac{2}{n}\operatorname{Re}\left\{\overline{b_{n,j}}\sum_{k=1}^n W_n\left(-\theta,\frac{k}{n}\right)e^{ik\left(2\theta+\frac{2\pi j}{n}\right)}\right\}$

$\frac{8\sin^2(\theta)}{n^2}\mathbb{I}_Y(\omega_j) = \left|\frac{1}{n}\sum_{k=1}^n W_n\left(-\theta,\frac{k}{n}\right)e^{-ik\frac{2\pi j}{n}}\right|^2 + o_p(1) + R_{n,j}$

$\sum_{j=1}^m R_{n,j} \leq 2\sup|b_{n,j}| \times \frac{m}{n}\sum_{k=1}^n \left|W_n\left(-\theta,\frac{k}{n}\right)\right| = o_p(1)\times O_p(m) = o_p(m)$

The same argument yields $\sum_{j=1}^m |b_{n,j}|^2 = o_p(m)$. Now:

$$G_{n,j} = \frac{1}{n}\sum_{k=1}^n W_n\left(-\theta,\frac{k}{n}\right)e^{-2\pi ij\frac{k}{n}} - \int_0^1 W_n(-\theta,t)\,e^{-2\pi ijt}dt$$

$G_{n,j} = \sum_{k=1}^n \int_{(k-1)/n}^{k/n}\left\{W_n\left(-\theta,\frac{k}{n}\right)e^{-2\pi ij\frac{k}{n}} - W_n(-\theta,t)\,e^{-2\pi ijt}\right\}dt$

$|G_{n,j}| \leq \sum_{k=1}^n \int_{(k-1)/n}^{k/n}\left\{\left|W_n\left(-\theta,\frac{k}{n}\right) - W_n(-\theta,t)\right| + \left|e^{-2\pi ij\frac{k}{n}} - e^{-2\pi ijt}\right| W_n(-\theta,t)\right\}dt$

$|G_{n,j}| \leq \sum_{k=1}^n \int_{(k-1)/n}^{k/n}\left\{\sup_{|u-v|<1/n}|W_n(-\theta,u) - W_n(-\theta,v)| + \sup_{0\leq t\leq 1}|W_n(-\theta,t)| \times \frac{Cm}{n}\right\}dt$

$$|G_{n,j}| \leq \omega_n\left(\frac{1}{n}\right) + \sup_{0\leq t\leq 1}|W_n(-\theta,t)| \times \frac{Cm}{n}$$

$\omega_n(\delta)$ the modulus of continuity[10] of $W_n(-\theta,.)$. We define $\widetilde{W_n}(-\theta,t) = W_n(-\theta,t) + T_n(t)$ the element of $C[0,1]$ interpolated from $W_n(-\theta,t)$, with $\sup_{0\leq t\leq 1}|T_n(t)| = o_p(1)$, and $\widetilde{\omega}_n(\delta)$ its modulus of continuity. Obviously:

$$\left|\omega_n\left(\frac{1}{n}\right) - \widetilde{\omega}_n\left(\frac{1}{n}\right)\right| \leq 2\sup_{0\leq t\leq 1}|T_n(t)|$$

and because $\widetilde{W_n}(-\theta,.)$ is tight:

$\forall\varepsilon,\eta > 0$, $\exists\delta > 0$, n_0 such as $P[\widetilde{\omega}_n(\delta) > \eta] < \varepsilon$. Hence, for n large enough, $P\left[\widetilde{\omega}_n\left(\frac{1}{n}\right) > \eta\right] < \varepsilon$ because $\widetilde{\omega}_n(.)$ is an increasing function. Thus, $\widetilde{\omega}_n\left(\frac{1}{n}\right) = o_p(1)$.

From $\sup_{0\leq t\leq 1}|W_n(-\theta,t)| = O_p(1)$, we get:

$G_{n,j} = o_p(1) + O_p\left(\frac{m}{n}\right) = o_p(1)$ <u>uniformly in j</u> by assumption on m.

It yields:

$$\left|\frac{1}{n}\sum_{k=1}^n W_n\left(-\theta,\frac{k}{n}\right)e^{-ik\frac{2\pi j}{n}}\right|^2 = \left|\int_0^1 W_n(-\theta,t)\,e^{-2\pi ijt}dt\right|^2 + H_{n,j}$$

with:

$$\begin{aligned} H_{n,j} &\leq |G_{n,j}| \times \left\{|G_{n,j}| + 2\left(\left|\int_0^1 W_n(-\theta,t)\,e^{-2\pi ijt}dt\right| + \left|\frac{1}{n}\sum_{k=1}^n W_n\left(-\theta,\frac{k}{n}\right)e^{-ik\frac{2\pi j}{n}}\right|\right)\right\} \\ &\leq |G_{n,j}| \times \left\{|G_{n,j}| + 4\sup_{0\leq t\leq 1}|W_n(-\theta,t)|\right\} = o_p(1) \text{ unif. en } j. \end{aligned}$$

[10] $\omega_n(\delta) = \sup_{|x-y|<\delta}|W_n(-\theta,x) - W_n(-\theta,y)|$

So we have:

$$\frac{8\sin^2(\theta)}{n^2}\sum_{j=0}^{m}\mathbb{I}_Y(\omega_j)=\sum_{j=0}^{m}\left|\int_0^1 W_n(-\theta,t)\,e^{-2\pi ijt}dt\right|^2+o_p(m)$$

But $\left|\int_0^1 W_n(-\theta,t)\,e^{-2\pi ijt}dt\right|^2=\iint_{[0,1]^2}W_n(-\theta,u)\,\overline{W_n(-\theta,v)}e^{-i2\pi j(u-v)}dudv$

Thus $\sum_{j=0}^{m}\left|\int_0^1 W_n(-\theta,t)\,e^{-2\pi ijt}dt\right|^2=\iint W_n(-\theta,u)\,\overline{W_n(-\theta,v)}\Delta_{m+1}[2\pi(u-v)]\,dudv$

$=(m+1)\int_0^1|W_n(-\theta,u)|^2du+\underbrace{\iint_{u\neq v}W_n(-\theta,u)\,\overline{W_n(-\theta,v)}\Delta_{m+1}[2\pi(u-v)]\,dudv}_{K_{n,m}}$

We have:

$$|K_{n,m}|\leq\left(\sup_{0\leq t\leq 1}|W_n(-\theta,t)|\right)^2\times\iint_{u\neq v}|\Delta_{m+1}[2\pi(u-v)]|\,dudv$$

Now, let $0<\delta<1$ be fixed.

$$\iint_{u\neq v}|\Delta_{m+1}[2\pi(u-v)]|\,dudv=\left(\iint_{0<|u-v|<\delta}+\iint_{\delta<|u-v|}\right)|\Delta_{m+1}[2\pi(u-v)]|\,dudv$$

From (3), the second term is bounded by $\iint_{0<|u-v|<\delta}\frac{1}{\pi\delta}dudv=C^{ste}$. Now,

$$\begin{aligned}\iint_{0<|u-v|<\delta}|\Delta_{m+1}[2\pi(u-v)]|\,dudv&\leq\int_{u=0}^{1}\left(\int_{v=u-\delta}^{u+\delta}|\Delta_{m+1}[2\pi(u-v)]|\,dv\right)du\\&=\int_{-\delta}^{\delta}|\Delta_{m+1}(2\pi x)|\,dx\end{aligned}$$

We know that this term is $O(\log m)$. Thus we get:

$$\frac{8\sin^2(\theta)}{n^2}\sum_{j=0}^{m}\mathbb{I}_Y(\omega_j)=(m+1)\int_0^1|W_n(-\theta,u)|^2du+O_p(\log m)$$

We obtain similarly $\frac{8\sin^2(\theta)}{n^2}\sum_{j=-m}^{1}\mathbb{I}_Y(\omega_j)=m\int_0^1|W_n(-\theta,u)|^2du+O_p(\log m)$ and finally:

$$\frac{8\sin^2(\theta)}{n^2}\widehat{f(\theta)}=\int_0^1|W_n(-\theta,u)|^2du+O_p\left(\frac{\log m}{2m+1}\right)$$

An application of the continuous mapping theorem yields:

$$\frac{8\sin^2(\theta)}{n^2}\widehat{f(\theta)}\Rightarrow\sigma^2\left|C_X\left(e^{-i\theta}\right)\right|^2\times\int_0^1|\mathbb{W}_c(t)|^2dt \tag{48}$$

The remaining point to check is the correction for deterministic components. But:

$$\mathbb{J}_{\widetilde{Y}}(\omega_j)=\mathbb{J}_Y(\omega_j)+O_p(1)\ \text{ uniformly in } j$$

It yields:

$$\mathbb{I}_{\tilde{Y}}(\omega_j) = \mathbb{I}_Y(\omega_j) + O_p(1) + 2\operatorname{Re}\{\mathbb{J}_Y(\omega_j) \times O_p(1)\}$$

But $\frac{2\sqrt{2}\sin(\theta)}{n}|\mathbb{J}_Y(\omega_j)| \leq \frac{2}{n}\sum_{k=1}^{n}\left|W_n\left(-\theta, \frac{k}{n}\right)\right| = O_p(1)$ uniformly. in j.
Hence $\mathbb{I}_{\tilde{Y}}(\omega_j)/n^2 = \mathbb{I}_Y(\omega_j)/n^2 + O_p(n^{-1})$ uniformly. in j and the estimator built with $\mathbb{I}_{\tilde{Y}}(\omega_j)$ admits the same limit law than (46). In particular, the nullity of the term associated with $\omega_0 = \theta$ has no asymptotic incidence. The convergence of ξ_n^p results from theorem (8), part *ii)* and the continuous mapping theorem.
■

8.7 Proof of lemma 13

$Y_n(t, \theta_n) - Y_n(t, \theta) = [C_X(\theta_n, B) - C_X(\theta, B)]\,\varepsilon_{[nt]}$ and:

$$\mathbb{E}\left|Y_n(t, \theta_n) - Y_n(t, \theta)\right|^2 = \frac{\sigma^2}{2\pi}\int_{-\pi}^{\pi}\left|C_X(\theta_n, e^{-i\omega}) - C_X(\theta, e^{-i\omega})\right|^2 d\omega$$

From (4), $|C_X(\theta_n, e^{-i\omega}) - C_X(\theta, e^{-i\omega})| \leq \sum_{k=0}^{\infty}\sum_{j=1}^{\infty}\left|\Psi_{j+k}\left(1 - e^{ij(\theta-\theta_n)}\right)\right|$
which is bounded by:
$C\sum_{k=0}^{K}\sum_{j=1}^{J}\left|\sin\left(\frac{j(\theta-\theta_n)}{2}\right)\right| + \left(\sum_{k=K+1}^{\infty}\sum_{j=1}^{J} + \sum_{k=0}^{K}\sum_{j=J+1}^{\infty} + \sum_{k=K+1}^{\infty}\sum_{j=J+1}^{\infty}\right)|\Psi_{j+k}|$
$\leq CKJ^2e^{-1}(n) + \sum_{j=1}^{J}\sum_{k=j+K+1}^{\infty}|\Psi_k| + \sum_{k=0}^{K}\sum_{j=k+J+1}^{\infty}|\Psi_k| + \sum_{k=K+1}^{\infty}\sum_{j=J+k+1}^{\infty}|\Psi_j|$

By assumption, $\Psi_m = O\left(m^{-d}\right)$ for $d > 2$. We then obtain the following bound:
$CKJ^2e^{-1}(n) + C_1\sum_{j=1}^{J}(j+K+1)^{-d+1} + C_1\sum_{k=0}^{K}(k+J+1)^{-d+1} + \sum_{k=K+1}^{\infty}(J+k+1)^{-d+1}$
or $CKJ^2e^{-1}(n) + C_1J(K+2)^{-d+1} + C_1K(J+2)^{-d+1} + C_2(J+K+1)^{-d+2}$

We set $K = J = [\log e(n)]$. With these values we obtain:

$$\left|C_X(\theta_n, e^{-i\omega}) - C_X(\theta_n, e^{-i\omega})\right| \leq O\left(\frac{[\log e(n)]^3}{e(n)}\right) + o(1) = o(1)$$

uniformly in ω. Hence:
$\mathbb{E}\left|Y_n(t, \theta_n) - Y_n(t, \theta)\right|^2 = o(1)$ and $Y_n(t, \theta_n) - Y_n(t, \theta) = o_p(1)$ uniformly. in t.
If f is C^5, then $\Psi_m = O\left(m^{-d}\right)$ for all $d < 4$, and we may proceed directly with the classical bound $|e^{ix} - 1 - x| \leq Cx$ for all real x:

$$\begin{aligned}|C_X(\theta_n, e^{-i\omega}) - C_X(\theta, e^{-i\omega})| &\leq \left(\sum_{k=0}^{\infty}\sum_{j=k+1}^{\infty}(j-k)|\Psi_j|\right)|\theta - \theta_n| \\ &\leq Ce^{-1}(n)\sum_{k=0}^{\infty}\sum_{j=k+1}^{\infty} j|\Psi_j| \leq Ce^{-1}(n)\sum_{k=0}^{\infty}k^{-d+2}\end{aligned}$$

The last expression is $O(e^{-1}(n))$ if d is chosen such as $d > 3$.
■

8.8 Proof of theorem 16

We write $G_n(.)$ in place of $G_n(\theta, .)$ in order to shorten the notations. We first proceed with the convergence of the finite-dimensional distributions $[G_n(t_1), \ldots, G_n(t_k)]'$ for $-2\pi \leq t_1 < t_2 < \ldots < t_k \leq 2\pi$. If $k = 1$, this is a consequence of theorem 3. If

$k=2$, let $\theta_n = \theta + t_1/e(n)$ and $\theta_n^{'} = \theta + t_2/e(n)$. The proof is similar to that used for theorem 19. Let's consider $\left[S_n^{'}(t_1), S_n^{'}(t_2)\right]^{'} =$
$\sqrt{\frac{2}{n}}\left(\sum_{k=1}^n \cos(\theta_n k)\varepsilon_k, \sum_{k=1}^n \sin(\theta_n k)\varepsilon_k, \sum_{k=1}^n \cos(\theta_n^{'} k)\varepsilon_k, \sum_{k=1}^n \sin(\theta_n^{'} k)\varepsilon_k\right)^{'}$
For $(a,b,c,d) \in \mathbb{R}^4$ fixed, we define a martingale difference array $u_{n,i}$ (already considered in theorem 19) by the expression:

$$u_{n,k} = x_{n,k}\varepsilon_k \text{ with } x_{n,k} = \frac{a\cos(\theta_n k) + b\sin(\theta_n k) + c\cos\left(\theta_n^{'} k\right) + d\sin\left(\theta_n^{'} k\right)}{\sigma\sqrt{\sum_{k=1}^n y_{n,k}^2}}$$

and $y_{n,k} = a\cos(\theta_n k) + b\sin(\theta_n k) + c\cos\left(\theta_n^{'} k\right) + d\sin\left(\theta_n^{'} k\right)$.
We have to prove that $\sum_{k=1}^n u_{nk} \Rightarrow \mathrm{N}(0,1)$. In order to show that $\sum_{k=1}^n u_{n,k}^2 \xrightarrow{P} 1$ and $\max_{1\le k\le n} |u_{n,k}| \underset{n\to\infty}{\xrightarrow{P}} 0$, it is enough to prove that $\sum_{k=1}^n y_{n,k}^2 \sim Cn$ for some $C > 0$ (see theorem 19).

$$y_{n,k}^2 = y_{n,k,1}^2 + y_{n,k,2}^2 + 2y_{n,k,1}y_{n,k,2} + 2\left(a\cos(\theta_n k) + b\sin(\theta_n k)\right)\left(c\cos\left(\theta_n^{'} k\right) + c\sin\left(\theta_n^{'} k\right)\right)$$

with: $y_{n,k,1} = a\cos(\theta_n k) + b\sin(\theta_n k)$and $y_{n,k,2} = c\cos\left(\theta_n^{'} k\right) + c\sin\left(\theta_n^{'} k\right)$
We already proved that for $\theta \in \left]0,\pi\right[$, $\sum_{k=1}^n \left(y_{n,k,1}^2 + y_{n,k,2}^2\right) \sim \frac{a^2+b^2+c^2+d^2}{2} n$
Now, for $\theta = 0$, we have:

$$\begin{aligned}
\frac{1}{n}\sum_{k=1}^n y_{n,k,1}^2 &= b^2 + (a^2-b^2)\left(\frac{1}{2} + \frac{1}{2}\frac{\cos[(n+1)\theta_n]\sin[n\theta_n]}{n\sin[\theta_n]}\right) + ab\frac{\sin[(n+1)\theta_n]\sin[n\theta_n]}{n\sin[\theta_n]} \\
&\sim b^2 + (a^2-b^2)\left(\frac{1}{2} + \frac{1}{2}\frac{\cos[(n+1)t_1/e(n)]\sin[nt_1/e(n)]}{n\sin[t_1/e(n)]}\right) + ab\frac{\sin[(n+1)t_1/e(n)]\sin[nt_1/e(n)]}{n\sin[t_1/e(n)]} \\
&\sim a^2 \text{ if } n = o\left[e(n)\right] \\
&\sim a^2\left(\frac{1}{2} + \frac{\sin[2t_1]}{4t_1}\right) + b^2\left(\frac{1}{2} - \frac{\sin[2t_1]}{4t_1}\right) + ab\frac{\sin^2[t_1]}{t_1} \text{ if } n = e(n)
\end{aligned}$$

and for $\theta = \pi$, we have:
$\frac{1}{n}\sum_{k=1}^n y_{n,k,1}^2 \sim b^2 + (a^2-b^2)\left(\frac{1}{2} + \frac{1}{2}\frac{(-1)^{n+1}\cos[(n+1)t_1/e(n)](-1)^n\sin[nt_1/e(n)]}{-n\sin[t_1/e(n)]}\right)$ and the same result.
Next, we get:

$$\frac{1}{n}\sum_{k=0}^{n-1}\cos(\theta_n k)\cos\left(\theta_n^{'} k\right) = o(1) + \frac{1}{2n}\cos\left(n\frac{\left[\theta_n - \theta_n^{'}\right]}{2}\right)\frac{\sin\left(n\frac{\left[\theta_n-\theta_n^{'}\right]}{2}\right)}{\sin\left(\frac{\left[\theta_n-\theta_n^{'}\right]}{2}\right)}$$

- $n = o\left[e(n)\right]$ and $\theta \in \left]0,\pi\right[$

  $$\frac{1}{n}\sum_{k=0}^{n-1}\cos(\theta_n k)\cos\left(\theta_n^{'} k\right) = o(1) + \frac{1}{2n}\cos\left(n\frac{\left[\theta_n - \theta_n^{'}\right]}{2}\right)\frac{\sin\left(n\frac{\left[\theta_n-\theta_n^{'}\right]}{2}\right)}{\sin\left(\frac{\left[\theta_n-\theta_n^{'}\right]}{2}\right)}$$

  The second term converges to $1/2$ when n tends to infinity because $n\frac{\left[\theta_n-\theta_n^{'}\right]}{2} = o(1)$. Similarly:
  $\frac{2}{n}\sum_{k=0}^{n-1}\sin(\theta_n k)\sin\left(\theta_n^{'} k\right) = 1 + o(1)$

$\frac{2}{n}\sum_{k=0}^{n-1}\cos(\theta_n k)\sin\left(\theta_n^{'}k\right)+\frac{2}{n}\sum_{k=0}^{n-1}\sin(\theta_n k)\cos\left(\theta_n^{'}k\right)=o(1)$

Finally $\sum_{k=1}^{n} y_{n,k}^2 \sim \left(\frac{a^2+b^2+c^2+d^2}{2}+ac+bd\right)\times n$.

The Central Limit Theorem for martingale difference sequences (Davidson (1994)) yields:

$(a,b,c,d)\left[S_n^{'}(t_1),S_n^{'}(t_2)\right]^{'} \Rightarrow \sigma\sqrt{(a+c)^2+(b+d)^2}\,\mathrm{N}(0,1)$

Define now $\mathbb{L}(t)=\mathbb{L}\overset{d}{\equiv}\mathrm{N}(0,\mathbb{I}_2)$and $X=\left(\mathbb{L}^{'},\mathbb{L}^{'}\right)^{'}$.

From the equality in distribution $(a,b,c,d)\,X \overset{d}{\equiv} \sqrt{(a+c)^2+(b+d)^2}\,\mathrm{N}(0,1)$, we get:

$$\left[S_n^{'}(t_1),S_n^{'}(t_2)\right]^{'} \Rightarrow \sigma\left(\mathbb{L}^{'},\mathbb{L}^{'}\right)^{'} \tag{49}$$

- $\underline{n=o[e(n)] \text{ and } \theta\in\{0,\pi\}}$

$\frac{2}{n}\sum_{k=0}^{n-1}\cos(\theta_n k)\cos\left(\theta_n^{'}k\right)=2+o(1)$

$\frac{2}{n}\sum_{k=0}^{n-1}\sin(\theta_n k)\sin\left(\theta_n^{'}k\right)=o(1)$

$\frac{2}{n}\sum_{k=0}^{n-1}\cos(\theta_n k)\sin\left(\theta_n^{'}k\right)+\frac{2}{n}\sum_{k=0}^{n-1}\sin(\theta_n k)\cos\left(\theta_n^{'}k\right)=o(1)$

and $\sum_{k=1}^{n} y_{n,k}^2 \sim (a^2+c^2+2ac)\times n$

From $(a,b,c,d)\,X \overset{d}{\equiv} \sqrt{2(a+c)^2}\,\mathrm{N}(0,1)$, we define:

$\left[S_n^{c}(t_1),S_n^{c}(t_2)\right]^{'}=\sqrt{\frac{2}{n}}\left(\sum_{k=1}^{n}\cos(\theta_n k)\varepsilon_k,\sum_{k=1}^{n}\cos(\theta_n^{'}k)\varepsilon_k\right)^{'}$

$\left[S_n^{s}(t_1),S_n^{s}(t_2)\right]^{'}=\sqrt{\frac{2}{n}}\left(\sum_{k=1}^{n}\sin(\theta_n k)\varepsilon_k,\sum_{k=1}^{n}\sin(\theta_n^{'}k)\varepsilon_k\right)^{'}$

and $\mathbb{L}(t)=\mathbb{L}\overset{d}{\equiv}\mathrm{N}(0,1)$. It is easy to verify that:

$$\begin{array}{l}\left[S_n^{c}(t_1),S_n^{c}(t_2)\right]^{'} \Rightarrow \sigma\sqrt{2}\,(\mathbb{L},\mathbb{L})^{'}\\ \left[S_n^{s}(t_1),S_n^{s}(t_2)\right]^{'} \Rightarrow 0\end{array} \tag{50}$$

and point $ii)$ is proved.

- $\underline{e(n)=n \text{ and } \theta\in]0,\pi[}$

$\frac{2}{n}\sum_{k=0}^{n-1}\cos(\theta_n k)\cos\left(\theta_n^{'}k\right)=o(1)+\frac{\sin(t_1-t_2)}{t_1-t_2}$

$\frac{2}{n}\sum_{k=0}^{n-1}\sin(\theta_n k)\sin\left(\theta_n^{'}k\right)=o(1)+\frac{\sin(t_1-t_2)}{t_1-t_2}$

$\frac{2}{n}\sum_{k=0}^{n-1}\cos(\theta_n k)\sin\left(\theta_n^{'}k\right)=o(1)-\frac{2\sin^2\left(\frac{t_1-t_2}{2}\right)}{t_1-t_2}$

$\frac{2}{n}\sum_{k=0}^{n-1}\sin(\theta_n k)\cos\left(\theta_n^{'}k\right)=o(1)+\frac{2\sin^2\left(\frac{t_1-t_2}{2}\right)}{t_1-t_2}$

and then $\sum_{k=1}^{n} y_{n,k}^2 \sim v(a,b,c,d)\times\frac{n}{2}$ with:

$v(a,b,c,d)=a^2+b^2+c^2+d^2+[ac+bd]\times\frac{2\sin(t_1-t_2)}{t_1-t_2}+[bc-ad]\times\frac{4\sin^2\left(\frac{t_1-t_2}{2}\right)}{t_1-t_2}$

This can be written as:

$\frac{1}{\sqrt{k(a,b,c,d)}}\left[S_n^{'}(t_1),S_n^{'}(t_2)\right]^{'} \Rightarrow \mathrm{N}(0,1)$

We now define $[\mathbb{L}(t)]_{0\le t\le 1}=[\mathbb{L}_1(t),\mathbb{L}_2(t)]^{'}$ as a gaussian stationary bivariate process with covariance function $\Gamma(h) = \mathbb{E}\mathbb{L}(t)\mathbb{L}(t+h)^{'}$:

$$\Gamma(h) = \begin{bmatrix} \frac{\sin(h)}{h} & \frac{2\sin^2(h/2)}{h} \\ -\frac{2\sin^2(h/2)}{h} & \frac{\sin(h)}{h} \end{bmatrix}, \ h \in [-4\pi, 4\pi] \tag{51}$$

(by a continuity argument, $\left(\frac{\sin(h)}{h}, \frac{\sin^2(h/2)}{h}\right)$ takes the value $(1,0)$ for $h=0$). It is now clear that:
$\sqrt{k(a,b,c,d)}\,\mathrm{N}(0,1) \overset{d}{\equiv} (a,b,c,d)\left[\mathbb{L}(t_1)^{'}, \mathbb{L}(t_2)^{'}\right]^{'}$. Hence,

$$\left[S_n^{'}(t_1), S_n^{'}(t_2)\right]^{'} \Rightarrow \sigma\left[\mathbb{L}(t_1)^{'}, \mathbb{L}(t_2)^{'}\right]^{'} \tag{52}$$

- $\underline{e(n) = n \text{ and } \theta \in \{0,\pi\}}$
$\frac{2}{n}\sum_{k=0}^{n-1}\cos(\theta_n k)\cos\left(\theta_n^{'}k\right) = o(1) + \frac{\sin(t_1+t_2)}{t_1+t_2} + \frac{\sin(t_1-t_2)}{t_1-t_2}$
$\frac{2}{n}\sum_{k=0}^{n-1}\sin(\theta_n k)\sin\left(\theta_n^{'}k\right) = o(1) - \frac{\sin(t_1+t_2)}{t_1+t_2} + \frac{\sin(t_1-t_2)}{t_1-t_2}$
$\frac{2}{n}\sum_{k=0}^{n-1}\cos(\theta_n k)\sin\left(\theta_n^{'}k\right) = o(1) + \frac{2\sin^2\left(\frac{t_1+t_2}{2}\right)}{t_1+t_2} - \frac{2\sin^2\left(\frac{t_1-t_2}{2}\right)}{t_1-t_2}$
$\frac{2}{n}\sum_{k=0}^{n-1}\sin(\theta_n k)\cos\left(\theta_n^{'}k\right) = o(1) + \frac{2\sin^2\left(\frac{t_1+t_2}{2}\right)}{t_1+t_2} + \frac{2\sin^2\left(\frac{t_1-t_2}{2}\right)}{t_1-t_2}$
With the notations of the previous point:
$$\begin{aligned} v(a,b,c,d) &= a^2\left(1+\frac{\sin[2t_1]}{2t_1}\right) + b^2\left(1-\frac{\sin[2t_1]}{2t_1}\right) + 2ab\frac{\sin^2[t_1]}{t_1} \\ &+c^2\left(1+\frac{\sin[2t_2]}{2t_2}\right) + d^2\left(1-\frac{\sin[2t_2]}{2t_2}\right) + 2cd\frac{\sin^2[t_2]}{t_2} \\ &+ac\times\left[\frac{\sin(t_1-t_2)}{t_1-t_2} + \frac{\sin(t_1+t_2)}{t_1+t_2}\right] + bd\times\left[\frac{\sin(t_1-t_2)}{t_1-t_2} - \frac{\sin(t_1+t_2)}{t_1+t_2}\right] \\ &+ad\times\left[\frac{2\sin^2\left(\frac{t_1+t_2}{2}\right)}{t_1+t_2} - \frac{2\sin^2\left(\frac{t_1-t_2}{2}\right)}{t_1-t_2}\right] + bc\times\left[\frac{2\sin^2\left(\frac{t_1+t_2}{2}\right)}{t_1+t_2} + \frac{2\sin^2\left(\frac{t_1-t_2}{2}\right)}{t_1-t_2}\right] \end{aligned}$$
We define now $[\mathbb{L}(t)]_{0\le t\le 1} = [\mathbb{L}_1(t), \mathbb{L}_2(t)]^{'}$ bivariate <u>non stationary</u> gaussian process with covariance function $\Gamma(t_1,t_2) = \mathbb{E}\mathbb{L}(t_1)\mathbb{L}(t_2)^{'}$. It is easy to verify that $\sqrt{k(a,b,c,d)}\,\mathrm{N}(0,1) \overset{d}{\equiv} (a,b,c,d)\left[\mathbb{L}(t_1)^{'}, \mathbb{L}(t_2)^{'}\right]^{'}$ when:

$$\Gamma(t_1,t_2) = \begin{bmatrix} \frac{\sin(t_1-t_2)}{t_1-t_2} + \frac{\sin(t_1+t_2)}{t_1+t_2} & \frac{2\sin^2\left(\frac{t_1+t_2}{2}\right)}{t_1+t_2} - \frac{2\sin^2\left(\frac{t_1-t_2}{2}\right)}{t_1-t_2} \\ \frac{2\sin^2\left(\frac{t_1+t_2}{2}\right)}{t_1+t_2} + \frac{2\sin^2\left(\frac{t_1-t_2}{2}\right)}{t_1-t_2} & \frac{\sin(t_1-t_2)}{t_1-t_2} - \frac{\sin(t_1+t_2)}{t_1+t_2} \end{bmatrix}$$

The proof for $k>2$ is omitted, because very similar. We turn now to tightness of the sequence $G_n(t_2)$.
$G_n(t_2) - G_n(t_1) = \sqrt{\frac{2}{n}}\left[\sum_{k=1}^{n}\left(e^{-ik\left(\theta+\frac{t_1}{e(n)}\right)} - e^{-ik\left(\theta+\frac{t_2}{e(n)}\right)}\right)\varepsilon_k\right]$
$$\begin{aligned} \mathbb{E}\left|G_n(t_2) - G_n(t_1)\right|^2 &= \frac{2\sigma^2}{n}\sum_{k=1}^{n}\left|e^{-ik\left(\theta+\frac{t_1}{e(n)}\right)} - e^{-ik\left(\theta+\frac{t_2}{e(n)}\right)}\right|^2 \\ &= \frac{2\sigma^2}{n}\sum_{k=1}^{n}\sin^2\left(k\frac{t_1-t_2}{e(n)}\right) \\ &\le \frac{\sigma^2}{2ne^2(n)}\left|t_1-t_2\right|^2\sum_{k=1}^{n}k^2 = O\left(\frac{n^2}{e^2(n)}\right)\left|t_1-t_2\right|^2 \end{aligned}$$
because $|\sin x| \le |x|$ for all x.

In the two cases under investigation, $n = O[e(n)]$. Hence, $\mathbb{E}|G_n(t_2) - G_n(t_1)|^2 \leq C|t_1 - t_2|^2$ for some $C > 0$. An appeal to theorem 12-3 of Billingsley (1968) yields the desired result. The assertion $i)$ of the theorem follows.
In complex notation, we get, when $e(n) = n$ and $\theta \in]0, \pi[$:
$G_n(t) \Rightarrow \sigma\mathbb{L}_c(t)$ with $\mathbb{L}_c(t) = \mathbb{L}_1(t) + i\mathbb{L}_2(t)$ complex gaussian process with covariance function:

$$r(h) = \mathbb{E}[\mathbb{L}_1(t) + i\mathbb{L}_2(t)][\mathbb{L}_1(t+h) - i\mathbb{L}_2(t+h)] = 2\frac{\sin(h)}{h} - 4i\frac{\sin^2(h/2)}{h}$$

In particular, $r(0) = 2$ and $G_c(t) \overset{d}{\equiv} \mathbb{W}_c(1)$.
When $e(n) = n$ and $\theta = 0$ or π, $r(t_1, t_2) = 2\frac{\sin(t_1-t_2)}{t_1-t_2} + 4i\frac{\sin^2[(t_1-t_2)/2]}{t_1-t_2} = r(t_2 - t_1)$: the complex process is stationary (with the same covariance function as before), but non-gaussian: point $iii)$ is proved.
Proof of point $iv)$ follows closely proof of theorem 19, by remarking that all the terms $\frac{2}{n}\sum_{j=0}^{n-1} h(j\theta_n)\, k\left(j\widetilde{\theta}_n\right)$ are $o(1)$ with $h(.), k(.) \in \{\cos(.), \sin(.)\}$.
■

8.9 Proof of theorem 19

We adopt the following representation for $S_n(t, \theta_n)$:

$$S_n(t,\theta_n) = \sqrt{\frac{2}{n}}\left(\sum_{k=1}^{[nt]}\cos(\theta_n k)\varepsilon_k, \sum_{k=1}^{[nt]}\sin(\theta_n k)\varepsilon_k, \sum_{k=1}^{[nt]}\cos(\widetilde{\theta}_n k)\varepsilon_k, \sum_{k=1}^{[nt]}\sin(\widetilde{\theta}_n k)\varepsilon_k\right)'$$

By an argument routinely used in the previous results of this paper (see for example theorem 8), we fix $(a, b, c, d) \in \mathbb{R}^4$, and we define:

$$u_{n,k} = x_{n,k}\varepsilon_k \text{ with } x_{n,i} = \frac{a\cos(\theta_n k) + b\sin(\theta_n k) + c\cos\left(\widetilde{\theta}_n k\right) + d\sin\left(\widetilde{\theta}_n k\right)}{\sigma\sqrt{\sum_{k=1}^{n}\left[a\cos(\theta_n k) + b\sin(\theta_n k) + c\cos\left(\widetilde{\theta}_n k\right) + d\sin\left(\widetilde{\theta}_n k\right)\right]^2}}$$

Let $y_{n,k} = a\cos(\theta_n k) + b\sin(\theta_n k) + c\cos\left(\widetilde{\theta}_n k\right) + d\sin\left(\widetilde{\theta}_n k\right)$.
The key-argument in the proof of $\sum_{k=1}^{n} u_{n,k}^2 \overset{P}{\to} 1$ is:

$$\sum_{i=1}^{n} y_{n,i}^2 \sim \frac{a^2+b^2+c^2+d^2}{2}n$$

Indeed, $y_{n,k}^2 = y_{n,k,1}^2 + y_{n,k,2}^2 + 2[a\cos(\theta_n k) + b\sin(\theta_n k)]\left[c\cos\left(\widetilde{\theta}_n k\right) + d\sin\left(\widetilde{\theta}_n k\right)\right]$
with: $y_{n,k,1} = a\cos(\theta_n k) + b\sin(\theta_n k)$ and $y_{n,k,2} = c\cos\left(\widetilde{\theta}_n k\right) + d\sin\left(\widetilde{\theta}_n k\right)$
From previous results, $\sum_{k=1}^{n}\left(y_{n,k,1}^2 + y_{n,k,2}^2\right) \sim \frac{a^2+b^2+c^2+d^2}{2}n$
Let's examine the remainder terms:

$\sum_{k=0}^{n-1}\cos(\theta_n k)\cos\left(\widetilde{\theta}_n k\right) = \cos\left(n\frac{\left[\theta_n+\widetilde{\theta}_n\right]}{2}\right)\frac{\sin\left(n\frac{\left[\theta_n+\widetilde{\theta}_n\right]}{2}\right)}{\sin\left(\frac{\left[\theta_n+\widetilde{\theta}_n\right]}{2}\right)}+\cos\left(n\frac{\left[\theta_n-\widetilde{\theta}_n\right]}{2}\right)\frac{\sin\left(n\frac{\left[\theta_n-\widetilde{\theta}_n\right]}{2}\right)}{\sin\left(\frac{\left[\theta_n-\widetilde{\theta}_n\right]}{2}\right)}$

The first term is bounded by $\left|\sin\left(\frac{\theta_n+\widetilde{\theta}_n}{2}\right)\right|^{-1}$ which $O(1)$ if $\theta<\pi$. If $\theta=\pi$, then:

$$\begin{aligned}\left(n\left|\sin\left(\tfrac{\theta_n+\widetilde{\theta}_n}{2}\right)\right|\right)^{-1} &\sim 2\tfrac{\widetilde{e}(n)}{n}\left|1+\tfrac{\widetilde{e}(n)}{e(n)}\right|^{-1}\\ &= o(1) \text{ by assumption}\end{aligned}$$

Moreover, in all cases:

$$\begin{aligned}\left(n\left|\sin\left(\tfrac{\theta_n-\widetilde{\theta}_n}{2}\right)\right|\right)^{-1} &\sim \left(n\left|\tfrac{\theta_n-\widetilde{\theta}_n}{2}\right|\right)^{-1} = 2\tfrac{\widetilde{e}(n)}{n}\left|1-\tfrac{\widetilde{e}(n)}{e(n)}\right|^{-1}\\ &= o(1)\times O(1) = o(1) \text{ if } \widetilde{e}(n) = o\left[e(n)\right]\end{aligned}$$

If $\widetilde{e}(n) = e(n)[1+\lambda_n]$ with $\lambda_n = o(1)$ and $\widetilde{e}(n) = o(n\lambda_n)$, we immediately have:

$$2\frac{\widetilde{e}(n)}{n}\left|1-\frac{\widetilde{e}(n)}{e(n)}\right|^{-1} = 2\frac{\widetilde{e}(n)}{n\lambda_n} = o(1)$$

and then $\sum_{k=0}^{n-1}\cos(\theta_n k)\cos\left(\widetilde{\theta}_n k\right) = o(n)$.
The other cross-products in the second term appearing in $y_{n,k}^2$ are handled in a similar way. The last step is $\max\limits_{1\le k\le n}|u_{n,k}| \underset{n\longrightarrow\infty}{\longrightarrow} 0$ which follows one more time from
$|u_{n,k}| \le \frac{(|a|+|b|+|c|+|d|)}{\sigma\sqrt{\sum_{k=1}^n y_{n,k}^2}}|\varepsilon_k| \le \frac{C}{\sqrt{n}}|\varepsilon_k|$
We finally get:

$$(a,b,c,d)\,T_n(t,\theta_n) \Rightarrow \sigma\sqrt{a^2+b^2+c^2+d^2}\mathbb{B}(t)$$

with $\mathbb{B}(t)$ standard Brownian motion, which is equivalent to the assertion of the theorem.
■

8.10 Proof of lemma 20

Assume that $\theta\neq\pi$, and let $\mathbb{J}_{\widetilde{u}^*}(\omega_j,t)$ be the DFT calculated from the residual $\widetilde{u}_{t,n}^*$of the regression of $S_t\left(\widehat{X},\theta\right)$ on $\sin t\theta$ and $\sin(t+1)\theta$, which uses the residuals of the first step regression $\widehat{X}_t$. We have only to prove that:

$$\left\|\mathbb{J}_{\widetilde{u}^*}(\omega_j,t)-\mathbb{J}_{\widetilde{u}}(\omega_j,t)\right\|_2 = O\left(n^{-1}\right) \text{ uniformly in } t$$

for $\omega_j=\frac{2\pi j}{n}$, $|j|\le m=o(n)$.
We get after some calculations:

$$S_t\left(\widehat{X},\theta\right) = S_t(X,\theta) - 2\,\mathrm{Re}\left[(\widehat{c}-c)\,S_t\left(e^{ivt},\theta\right)\right]$$

with

$$S_t\left(e^{ivt},\theta\right) = \frac{2ie^{iv}}{\sin\theta}\left\{\frac{e^{it\theta}-e^{it\nu}}{1-e^{i(v-\theta)}} - \frac{e^{-it\theta}-e^{it\nu}}{1-e^{i(v+\theta)}}\right\}$$

Let $w_t = u_t - 2\,\mathrm{Re}\left[(\widehat{c} - c)\, S_t\left(e^{ivt}, \theta\right)\right]$. As in lemma 9:

$$\mathbb{J}_{\widetilde{u}^*}(\omega_j, t) - \mathbb{J}_w(\omega_j, t) = -\frac{1}{n}\left(1 - \frac{|\Delta_n(2\theta)|^2}{n^2}\right)^{-1} \times \left(\Delta_n(-\theta - \omega_j)\left\{\overline{\mathbb{J}_w(\theta, t)} - \frac{\Delta_n(2\theta)}{n}\mathbb{J}_w(\theta, t)\right\}\right) \quad (53)$$

We consider only the case $v = \theta$. Then $S_t\left(e^{ivt}, \theta\right) = -\frac{2i \sin t\theta}{\sin^2 \theta}$

$$\mathbb{J}_w(\omega, t) - \mathbb{J}_u(\omega, t) = C \times \mathrm{Im}(\widehat{c} - c)\left[\Delta_n(\theta + \omega) - \Delta_n(\theta - \omega)\right]$$

Then $\left\|\mathbb{J}_w(\omega_j, t) - \mathbb{J}_u(\omega_j, t)\right\|_2 = C \times \left\|\mathrm{Im}(\widehat{c} - c)\, \Delta_n(\theta + \omega_j)\right\|_2 = O\left(n^{-1}\right)$ under $\mathbf{H}_0$. This result and (**??**) yields:

$$\left\|\mathbb{J}_{\widetilde{u}^*}(\omega_j, t) - \mathbb{J}_u(\omega_j, t)\right\|_2 = O\left(n^{-1}\right)$$

■

8.11 Tabulation of $\sup_{t \in [0,1]} |W_c(t)|$

We consider $X_t^i = \varepsilon_t^i$ with ε_t^i drawn independently in $\mathrm{N}(0, 1)$ for $t = 1, \ldots, N_p$ and $i = 1, \ldots, N_s$, N_s being the number of simulated samples and N_p the size of each sample. Let $\mathbb{I}_G$ be a finite grid partitioning $[0, 1]$: $\mathbb{I}_G = \left\{\frac{j}{N_p}, j = 0, 1, \ldots, N_p\right\}$.
The following approximation (in law) holds when N_p is large enough:
$Y_i = \sup_{t \in \mathbb{I}_G} \left|\mathbb{J}_X\left(\frac{\pi}{2}, t\right)\right| \backsimeq \frac{1}{\sqrt{2}} \sup_{t \in [0,1]} |\mathbb{W}_c(t)|$. Therefore, the quantiles of $\sup_{t \in [0,1]} |\mathbb{W}_c(t)|$ are obtained from the empirical law of the variables $2Y_i^2$. If $P\left(\sup_{t \in \mathbb{I}_G} |\mathbb{W}_c(t)|^2 > c_{1-\alpha}\right) = \alpha$ and $\widehat{c}_{1-\alpha} = c(N_s, \alpha) = F_{N_s}^{-1}(1 - \alpha)$ is the empirical quantile, we know that $\mathbb{I}_{N_s} = P\left(\sup_{t \in \mathbb{I}_G} |\mathbb{W}_c(t)|^2 > c(N_p, \alpha)\right)$ has expectation α and (approximately) variance $\widehat{\sigma}_{1-\alpha}^2 = \frac{\alpha(1-\alpha)}{N_s} \times h^{-2}(\widehat{c}_{1-\alpha})$ where $h(x)$ is the density of probability of $\sup_{t \in [0,1]} |\mathbb{W}_c(t)|^2$. The error on c_α is then quantified by $\widehat{\sigma}_\alpha$. With $N_s = 6000$ and $N_p = 500$, we obtain the following table for both quantiles and their standard errors.

α	0.01	0.025	0.05	0.10	0.15	0.50	0.85	0.90	0.95	0.975	0.99
$\widehat{c}_{1-\alpha}$	9.97	8.38	7.01	5.65	4.97	1.537	2.46	0.87	0.64	0.41	0.19
$\widehat{\sigma}_{1-\alpha}$	0.20	0.15	0.10	0.07	0.06	0.03	0.02	0.02	0.02	0.02	0.04

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